c. From Exercise 5, the production $\mathbf{x}$ corressponding to $\mathbf{d}=\begin{bmatrix}50\\20\end{bmatrix}$ is $\mathbf{x}=\begin{bmatrix}110\\120\end{bmatrix}$.

Note that $\mathbf{d}_2=\mathbf{d}+\mathbf{d}_1$. Thus

$$\begin{aligned}\mathbf{x}_2&=(I-C)^{-1}\mathbf{d}_2\\&=(I-C)^{-1}(\mathbf{d}+\mathbf{d}_1)\\&=(I-C)^{-1}\mathbf{d}+(I-C)^{-1}\mathbf{d}_1\\&=\mathbf{x}+\mathbf{x}_1\end{aligned}$$

8. a. Given $(I-C)\mathbf{x}=\mathbf{d}$ and $(I-C)\Delta\mathbf{x}=\Delta\mathbf{d}$,

$$(I-C)(\mathbf{x}+\Delta\mathbf{x})=(I-C)\mathbf{x}+(I-C)\Delta\mathbf{x}=\mathbf{d}+\Delta\mathbf{d}$$

Thus $\mathbf{x}+\Delta\mathbf{x}$ is the production level corresponding to a demand of $\mathbf{d}+\Delta\mathbf{d}$.

b. Since $\Delta\mathbf{x}=(I-C)^{-1}\Delta\mathbf{d}$ and $\Delta\mathbf{d}$ is the first column of I, $\Delta\mathbf{x}$ will be the first column of $(I-C)^{-1}$.

9. In this case

$$I-C=\begin{bmatrix}.8&-.2&.0\\-.3&.9&-.3\\-.1&.0&.8\end{bmatrix}$$

Row reduce $[\,I-C\ \ \mathbf{d}\,]$ to find

$$\begin{bmatrix}.8&-.2&.0&40.0\\-.3&.9&-.3&60.0\\-.1&.0&.8&80.0\end{bmatrix}\sim\begin{bmatrix}1&0&0&82.8\\0&1&0&131.0\\0&0&1&110.3\end{bmatrix}$$

So $\mathbf{x}=(82.8,\ 131.0,\ 110.3)$.

10. From Exercise 8, the (i,j) entry in $(I-C)^{-1}$ corresponds to the effect on production of sector i when the final demand for the output of sector j increases by one unit. Since these entries are all positive, an increase in the final demand for any sector will cause the production of all sectors to increase. Thus an increase in the demand for any sector will lead to an increase in the demand for all sectors.

11. (Solution in *study Guide*) Following the hint in the text, compute $\mathbf{p}^T\mathbf{x}$ in two ways. First, take the transpose of both sides of the price equation, $\mathbf{p}=C^T\mathbf{p}+\mathbf{v}$, to obtain

$$\mathbf{p}^T=(C^T\mathbf{p}+\mathbf{v})^T=(C^T\mathbf{p})^T+\mathbf{v}^T=\mathbf{p}^TC+\mathbf{v}^T$$

and right-multiply by $\mathbf{x}$ to get

$$\mathbf{p}^T\mathbf{x}=(\mathbf{p}^TC+\mathbf{v}^T)\mathbf{x}=\mathbf{p}^TC\mathbf{x}+\mathbf{v}^T\mathbf{x}$$

Another way to compute $\mathbf{p}^T\mathbf{x}$ starts with the production equation $\mathbf{x}=C\mathbf{x}+\mathbf{d}$. Left multiply by $\mathbf{p}^T$ to get

$$\mathbf{p}^T\mathbf{x}=\mathbf{p}^T(C\mathbf{x}+\mathbf{d})=\mathbf{p}^TC\mathbf{x}+\mathbf{p}^T\mathbf{d}$$

The two expression for $\mathbf{p}^T\mathbf{x}$ show that

$$\mathbf{p}^TC\mathbf{x}+\mathbf{v}^T\mathbf{x}=\mathbf{p}^TC\mathbf{x}+\mathbf{p}^T\mathbf{d}$$

so $\mathbf{v}^T\mathbf{x}=\mathbf{p}^T\mathbf{d}$. The *Study Guide* also provides a slightly different solution.

12. Since

1.9 SOLUTIONS

Notes: This section is optional if you plan to treat linear transformations only lightly, but many instructors will want to cover at least Theorem 10 and a few geometric examples. Exercises 15 and 16 illustrate a fast way to solve Exercises 17–22 without explicitly computing the images of the standard basis.

The purpose of introducing *one-to-one* and *onto* is to prepare for the term *isomorphism* (in Section 4.4) and to acquaint math majors with these terms. Mastery of these concepts would require a substantial digression, and some instructors prefer to omit these topics (and Exercises 25–40). In this case, you can use the result of Exercise 31 in Section 1.8 to show that the coordinate mapping from a vector space onto $\mathbf{R}^n$ (in Section 4.4) preserves linear independence and dependence of sets of vectors. (See Example 6 in Section 4.4.) The notions of one-to-one and onto appear in the Invertible Matrix Theorem (Section 2.3), but can be omitted there if desired.

Exercises 25–28 and 31–36 offer fairly easy writing practice. Exercises 31, 32, and 35 provide important links to earlier material.

1. $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$

2. $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$

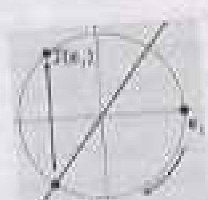
3. $T(\mathbf{e}_1) = -\mathbf{e}_2,\ T(\mathbf{e}_2) = \mathbf{e}_1,\ A = [-\mathbf{e}_2 \quad \mathbf{e}_1] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

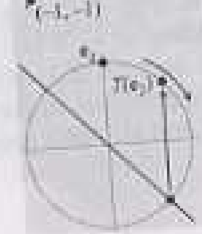
4. $T(\mathbf{e}_1) = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix},\ T(\mathbf{e}_2) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

5. $T(\mathbf{e}_1) = \mathbf{e}_1 - 2\mathbf{e}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ T(\mathbf{e}_2) = \mathbf{e}_2,\ A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

6. $T(\mathbf{e}_1) = \mathbf{e}_1,\ T(\mathbf{e}_2) = \mathbf{e}_2 + 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix},\ A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

7. Follow what happens to $\mathbf{e}_1$ and $\mathbf{e}_2$. Since $\mathbf{e}_1$ is on the unit circle in the plane, it rotates through $-3\pi/4$ radians into a point on the unit circle that lies in the third quadrant and on the line $x_2 = x_1$ (that is, $y = x$ in more familiar notation). The point $(-1,-1)$ is on the ine $x_2 = x_1$, but its distance from the origin is $\sqrt{2}$. So the rotational image of $\mathbf{e}_1$ is $(-1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(-1/\sqrt{2}, 1/\sqrt{2})$.





Similarly, $\mathbf{e}_2$ rotates into a point on the unit circle that lies in the second quadrant and on the line $x_2 = x_1$, namely,

$(-1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(-1/\sqrt{2}, 1/\sqrt{2})$.

When the two calculations described above are written in vertical vector notation, the transformation's standard matrix $[T(\mathbf{e}_1) \quad T(\mathbf{e}_2)]$ is easily seen:

$$\mathbf{e}_1 \to \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{e}_2 \to \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

8. $\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$ and $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$, so $A = [\mathbf{e}_2 \quad -\mathbf{e}_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

9. The horizontal shear maps $\mathbf{e}_1$ into $\mathbf{e}_1$, and then the reflection in the line $x_2 = -x_1$ maps $\mathbf{e}_1$ into $-\mathbf{e}_2$. (See Table 1.) The horizontal shear maps $\mathbf{e}_2$ into $\mathbf{e}_2$ into $\mathbf{e}_2 - 2\mathbf{e}_1$. To find the image of $\mathbf{e}_2 - 2\mathbf{e}_1$ when it is reflected in the line $x_2 = -x_1$, use the fact that such a reflection is a linear transformation. So, the image of $\mathbf{e}_2 - 2\mathbf{e}_1$ is the same linear combination of the images of $\mathbf{e}_2$ and $\mathbf{e}_1$, namely, $-\mathbf{e}_1 - 2(-\mathbf{e}_2) = -\mathbf{e}_1 + 2\mathbf{e}_2$. To summarize,

$$\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_2 \text{ and } \mathbf{e}_2 \to \mathbf{e}_2 - 2\mathbf{e}_1 \to -\mathbf{e}_1 + 2\mathbf{e}_2, \text{ so } A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

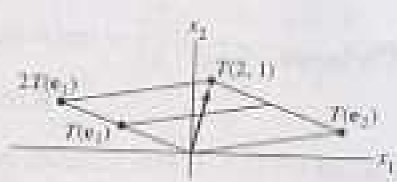
To find the image of $\mathbf{e}_2 - 2\mathbf{e}_1$ when it is reflected through the vertical axis use the fact that such a reflection is a linear transformation. So, the image of $\mathbf{e}_2 - 2\mathbf{e}_1$ is the same linear combination of the images of $\mathbf{e}_2$ and $\mathbf{e}_1$, namely, $\mathbf{e}_2 + 2\mathbf{e}_1$.

10. $\mathbf{e}_1 \to -\mathbf{e}_1 \to -\mathbf{e}_2$ and $\mathbf{e}_2 \to \mathbf{e}_2 \to -\mathbf{e}_1$, so $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

11. The transformation T described maps $\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_1$ and maps $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_2$. A rotation through π radians also maps $\mathbf{e}_1$ into $-\mathbf{e}_1$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_2$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

12. The transformation T in Exercise 8 maps $\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$ and maps $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$. A rotation about the origin through $\pi/2$ radians also maps $\mathbf{e}_1$ into $\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_1$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

13. Since $(2, 1) = 2\mathbf{e}_1 + \mathbf{e}_2$, the image of $(2, 1)$ under T is $2T(\mathbf{e}_1) + T(\mathbf{e}_2)$, by linearity of T. On the figure in the exercise, locate $2T(\mathbf{e}_1)$ and use it with $T(\mathbf{e}_2)$ to form the parallelogram shown below.



9. For $\lambda = 1$: $A - 1I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$

The augmented matrix for $(A - I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. Thus $x_1 = 0$ and x_2 is free. The general solution of $(A - I)\mathbf{x} = \mathbf{0}$ is $x_2\mathbf{e}_2$, where $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and so $\mathbf{e}_2$ is a basis for the eigenspace corresponding to the eigenvalue 1.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}$

The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ leads to $2x_1 - 4x_2 = 0$, so that $x_1 = 2x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace.

10. For $\lambda = 4$: $A - 4I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$.

The augmented matrix for $(A - 4I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9/6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (3/2)x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (3/2)x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to 4 is $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

11. $A - 10I = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix}$

The augmented matrix for $(A - 10I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -6 & -2 & 0 \\ -3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (-1/3)x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(1/3)x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to 10 is $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

12. For $\lambda = 1$: $A - I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$

The augmented matrix for $(A - I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (-2/3)x_2$ and x_2 is free. A basis for the eigenspace corresponding to 1 is $\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}$.

The augmented matrix for $(A-5I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = 2x_2$ and x_2 is

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. A basis for the eigenspace is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

13. For $\lambda = 1$:

$$A-1I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

The equations for $(A-I)\mathbf{x}=\mathbf{0}$ are easy to solve: $\begin{cases} 3x_1 + x_3 = 0 \\ -2x_1 \qquad = 0 \end{cases}$

Row operations hardly seem necessary. Obviously x_1 is zero, and hence x_3 is also zero. There are three-variables, so x_2 is free. The general solution of $(A-I)\mathbf{x}=\mathbf{0}$ is $x_2\mathbf{e}_2$, where $\mathbf{e}_2 = (0,1,0)$, and so $\mathbf{e}_2$ provides a basis for the eigenspace.

For $\lambda = 2$:

$$A-2I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$[(A-2I)\ \ \mathbf{0}] = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} ① & 0 & 1/2 & 0 \\ 0 & ① & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -(1/2)x_3$, $x_2 = x_3$, with x_3 free. The general solution of $(A-2I)\mathbf{x}=\mathbf{0}$ is $x_3\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$. A nice basis vector for the eigenspace is $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

For $\lambda = 3$:

$$A-3I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

$$[(A-3I)\ \ \mathbf{0}] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} ① & 0 & 1 & 0 \\ 0 & ① & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -x_3$, $x_2 = x_3$, with x_3 free. A basis vector for the eigenspace is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

17. The eigenvalues of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ are 0, 2, and -1, on the main diagonal, by Theorem 1.

18. The eigenvalues of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are 4, 0, and -3, on the main diagonal, by Theorem 1.

19. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of the matrix. See the discussion following Example 5.

20. The matrix $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of A. Eigenvectors for the eigenvalue 0 are solutions of $A\mathbf{x} = \mathbf{0}$ and therefore have entries that produce a linear dependence relation among the columns of A. Any nonzero vector (in $\mathbb{R}^3$) whose entries sum to 0 will work. Find any two such vectors that are not multiples; for instance, $(1, 1, -2)$ and $(1, -1, 0)$.

21. a. False. The equation $A\mathbf{x} = \lambda\mathbf{x}$ must have a *nontrivial* solution.

b. True. See the paragraph after Example 5.

c. True. See the discussion of equation (3).

d. True. See Example 2 and the paragraph preceding it. Also, see the Numerical Note.

e. False. See the warning after Example 3.

22. a. False. The vector $\mathbf{x}$ in $A\mathbf{x} = \lambda\mathbf{x}$ must be *nonzero*.

b. False. See Example 4 for a two-dimensional eigenspace, which contains two linearly independent eigenvectors corresponding to the same eigenvalue. The statement given is not at all the same as Theorem 2. In fact, it is the *converse* of Theorem 2 (for the case $r = 2$).

c. True. See the paragraph after Example 1.

d. False. Theorem 1 concerns a *triangular* matrix. See Examples 3 and 4 for counterexamples.

e. True. See the paragraph following Example 3. The eigenspace of A corresponding to λ is the null space of the matrix $A - \lambda I$.

23. If a 2×2 matrix A were to have three distinct eigenvalues, then by Theorem 2 there would correspond three linearly independent eigenvectors (one for each eigenvalue). This is impossible because the vectors all belong to a two-dimensional vector space, in which any set of three vectors is linearly dependent. See Theorem 8 in Section 1.7. In general, if an $n \times n$ matrix has p distinct eigenvalues, then by Theorem 2 there would be a linearly independent set of p eigenvectors (one for each eigenvalue). Since these vectors belong to an n-dimensional vector space, p cannot exceed n.

24. A simple example of a 2×2 matrix with only one distinct eigenvalue is a triangular matrix with the same number on the diagonal. By experimentation, one finds that if such a matrix is actually a diagonal matrix then the eigenspace is two dimensional, and otherwise the eigenspace is only one dimensional.

$\begin{bmatrix} 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 4 & 5 \end{bmatrix}$

25. If λ is an eigenvalue of A, then there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Since A is invertible, $A^{-1}A\mathbf{x} = A^{-1}(\lambda\mathbf{x})$, and so $\mathbf{x} = \lambda(A^{-1}\mathbf{x})$. Since $\mathbf{x} \neq \mathbf{0}$ (and since A is invertible), λ cannot be zero. Then $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that λ^{-1} is an eigenvalue of A^{-1}.

Note: The *Study Guide* points out here that the relation between the eigenvalues of A and A^{-1} is important in the so-called *inverse power method* for estimating an eigenvalue of a matrix. See Section 5.8.

26. Suppose that A^2 is the zero matrix. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$, then $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{x}$ is nonzero, λ must be nonzero. Thus each eigenvalue of A is zero.

27. Use the *Hint* in the text to write, for any λ, $(A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$. Since $(A - \lambda I)^T$ is invertible if and only if $A - \lambda I$ is invertible (by Theorem 6(c) in Section 2.2), it follows that $A^T - \lambda I$ is *not* invertible if and only if $A - \lambda I$ is *not* invertible. That is, λ is an eigenvalue of A^T if and only if λ is an eigenvalue of A.

Note: If you discuss Exercise 27, you might ask students on a test to show that A and A^T have the same characteristic polynomial (discussed in Section 5.2). Since $\det A = \det A^T$, for any square matrix A,

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - (\lambda I)^T) = \det(A - \lambda I).$$

28. If A is lower triangular, then A^T is upper triangular and has the same diagonal entries as A. Hence, by the part of Theorem 1 already proved in the text, these diagonal entries are eigenvalues of A^T. By Exercise 27, they are also eigenvalues of A.

29. Let $\mathbf{v}$ be the vector in $\mathbf{R}^n$ whose entries are all ones. Then $A\mathbf{v} = s\mathbf{v}$.

30. Suppose the column sums of an $n \times n$ matrix A all equal the same number s. By Exercise 29 applied to A^T in place of A, the number s is an eigenvalue of A^T. By Exercise 27, s is an eigenvalue of A.

31. Suppose T reflects points across (or through) a line that passes through the origin. That line consists of all multiples of some nonzero vector $\mathbf{v}$. The points on this line do not move under the action of A. So $T(\mathbf{v}) = \mathbf{v}$. If A is the standard matrix of T, then $A\mathbf{v} = \mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 1. The eigenspace is Span $\{\mathbf{v}\}$. Another eigenspace is generated by any nonzero vector $\mathbf{u}$ that is perpendicular to the given line. (Perpendicularity in $\mathbf{R}^2$ should be a familiar concept even though orthogonality in $\mathbf{R}^n$ has not been discussed yet.) Each vector $\mathbf{x}$ on the line through $\mathbf{u}$ is transformed into the vector $-\mathbf{x}$. The eigenvalue is -1.

33. (The solution is given in the text.)

a. Replace k by $k+1$ in the definition of $\mathbf{x}_k$, and obtain $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$.

b.
$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) \\ &= c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v} \quad \text{by linearity} \\ &= c_1\lambda^k \lambda\mathbf{u} + c_2\mu^k \mu\mathbf{v} \quad \text{since } \mathbf{u} \text{ and } \mathbf{v} \text{ are eigenvectors} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

34. You could try to write $\mathbf{x}_0$ as linear combination of eigenvectors, $\mathbf{v}_1, \ldots, \mathbf{v}_p$. If $\lambda_1, \ldots, \lambda_p$ are corresponding eigenvalues, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then you could *define*

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p$$

In this case, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p) \\ &= c_1\lambda_1^k A\mathbf{v}_1 + \cdots + c_p\lambda_p^k A\mathbf{v}_p \quad \text{Linearity} \\ &= c_1\lambda_1^{k+1}\mathbf{v}_1 + \cdots + c_p\lambda_p^{k+1}\mathbf{v}_p \quad \text{The } \mathbf{v}_i \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

35. Using the figure in the exercise, plot $T(\mathbf{u})$ as $2\mathbf{u}$, because $\mathbf{u}$ is an eigenvector for the eigenvalue 2 of the standard matrix A. Likewise, plot $T(\mathbf{v})$ as $3\mathbf{v}$, because $\mathbf{v}$ is an eigenvector for the eigenvalue 3. Since T is linear, the image of $\mathbf{w}$ is $T(\mathbf{w}) = T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

36. As in Exercise 35, $T(\mathbf{u}) = -\mathbf{u}$ and $T(\mathbf{v}) = 3\mathbf{v}$ because $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for the eigenvalues -1 and 3, respectively, of the standard matrix A. Since T is linear, the image of $\mathbf{w}$ is $T(\mathbf{w}) = T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

Note: The matrix programs supported by this text all have an eigenvalue command. In some cases, such as MATLAB, the command can be structured so it provides eigenvectors as well as a list of the eigenvalues. At this point in the course, students should *not* use the extra power that produces eigenvectors. Students need to be reminded frequently that eigenvectors of A are null vectors of a translate of A. That is why the instructions for Exercises 35–38 tell students to use the method of Example 4.

It is my experience that nearly all students need manual practice finding eigenvectors by the method of Example 4, at least in this section if not also in Sections 5.2 and 5.3. However, [M] exercises do create a burden if eigenvectors must be found manually. For this reason, the data files for the text include a special command, `nulbasis` for each matrix program (MATLAB, Maple, etc.). The output of `nulbasis (A)` is a matrix whose columns provide a basis for the null space of A, and these columns are identical to the ones a student would find by row reducing the augmented matrix $[A \;\; \mathbf{0}]$. With nulbasis, student answers will be the same (up to multiples) as those in the text. I encourage my students to use technology to speed up all numerical homework here, not just the **[M]** exercises.

37. **[M]** Let A be the given matrix. Use the MATLAB commands `eig` and `nulbasis` (or equivalent commands). The command `ev = eig(A)` computes the three eigenvalues of A and stores them in a vector `ev`. In this exercise, `ev` = (3, 13, 13). The eigenspace for the eigenvalue 3 is the null space of $A - 3I$. Use `nulbasis` to produce a basis for each null space. If the format is set for rational display, the result is

$$\texttt{nulbasis(A-ev(1)*eye(3))} = \begin{bmatrix} 5/9 \\ -2/9 \\ 1 \end{bmatrix}.$$

For simplicity, scale the entries by 9. A basis for the eigenspace for $\lambda = 3$: $\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

5.2 SOLUTIONS

Notes: Exercises 9–14 can be omitted, unless you want your students to have some facility with determinants of 3×3 matrices. In later sections, the text will provide eigenvalues when they are needed for matrices larger than 2×2. If you discussed partitioned matrices in Section 2.4, you might wish to bring in Supplementary Exercises 12–14 in Chapter 5. (Also, see Exercise 14 of Section 2.4.)

Exercises 25 and 27 support the subsection on dynamical systems. The calculations in these exercises and Example 5 prepare for the discussion in Section 5.6 about eigenvector decompositions.

1. $A=\begin{bmatrix}2&7\\7&2\end{bmatrix}, A-\lambda I=\begin{bmatrix}2&7\\7&2\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}=\begin{bmatrix}2-\lambda&7\\7&2-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(2-\lambda)^2-7^2=4-4\lambda+\lambda^2-49=\lambda^2-4\lambda-45$$

In factored form, the characteristic equation is $(\lambda-9)(\lambda+5)=0$, so the eigenvalues of A are 9 and −5.

2. $A=\begin{bmatrix}5&3\\3&5\end{bmatrix}, A-\lambda I=\begin{bmatrix}5-\lambda&3\\3&5-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(5-\lambda)(5-\lambda)-3\cdot 3=\lambda^2-10\lambda+16$$

Since $\lambda^2-10\lambda+16=(\lambda-8)(\lambda-2)$, the eigenvalues of A are 8 and 2.

3. $A=\begin{bmatrix}3&-2\\1&-1\end{bmatrix}, A-\lambda I=\begin{bmatrix}3-\lambda&-2\\1&-1-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(3-\lambda)(-1-\lambda)-(-2)(1)=\lambda^2-2\lambda-1$$

Use the quadratic formula to solve the characteristic equation and find the eigenvalues:

$$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}$$

4. $A=\begin{bmatrix}5&-3\\-4&3\end{bmatrix}, \ A-\lambda I=\begin{bmatrix}5-\lambda&-3\\-4&3-\lambda\end{bmatrix}$. The characteristic polynomial of A is

$$\det(A-\lambda I)=(5-\lambda)(3-\lambda)-(-3)(-4)=\lambda^2-8\lambda+3$$

Use the quadratic formula to solve the characteristic equation and find the eigenvalues:

$$\lambda=\frac{8\pm\sqrt{64-4(3)}}{2}=\frac{8\pm2\sqrt{13}}{2}=4\pm\sqrt{13}$$

5. $A=\begin{bmatrix}2&1\\-1&4\end{bmatrix}, A-\lambda I=\begin{bmatrix}2-\lambda&1\\-1&4-\lambda\end{bmatrix}$. The characteristic polynomial of A is

$$\det(A-\lambda I)=(2-\lambda)(4-\lambda)-(1)(-1)=\lambda^2-6\lambda+9=(\lambda-3)^2$$

Thus, A has only one eigenvalue 3, with multiplicity 2.

6. $A=\begin{bmatrix}3&-4\\4&8\end{bmatrix}, A-\lambda I=\begin{bmatrix}3-\lambda&-4\\4&8-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(3-\lambda)(8-\lambda)-(-4)(4)=\lambda^2-11\lambda+40$$

10. $\det(A-\lambda I)=\det\begin{bmatrix}0-\lambda & 3 & 1\\ 3 & 0-\lambda & 2\\ 1 & 2 & 0-\lambda\end{bmatrix}$. From the special formula for 3×3 determinants, the characteristic polynomial is

$$\det(A-\lambda I)=(-\lambda)(-\lambda)(-\lambda)+3\cdot2\cdot1+1\cdot3\cdot2-1\cdot(-\lambda)\cdot1-2\cdot2\cdot(-\lambda)-(-\lambda)\cdot3\cdot3$$
$$=-\lambda^3+6+6+\lambda+4\lambda+9\lambda=-\lambda^3+14\lambda+12$$

11. The special arrangements of zeros in A makes a cofactor expansion along the first row highly effective.

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & 0 & 0\\ 5 & 3-\lambda & 2\\ -2 & 0 & 2-\lambda\end{bmatrix}=(4-\lambda)\det\begin{bmatrix}3-\lambda & 2\\ 0 & 2-\lambda\end{bmatrix}$$
$$=(4-\lambda)(3-\lambda)(2-\lambda)=(4-\lambda)(\lambda^2-5\lambda+6)=-\lambda^3+9\lambda^2-26\lambda+24$$

If only the eigenvalues were required, there would be no need here to write the characteristic polynomial in expanded form.

12. Make a cofactor expansion along the third row:

$$\det(A-\lambda I)=\det\begin{bmatrix}-1-\lambda & 0 & 1\\ -3 & 4-\lambda & 1\\ 0 & 0 & 2-\lambda\end{bmatrix}=(2-\lambda)\cdot\det\begin{bmatrix}-1-\lambda & 0\\ -3 & 4-\lambda\end{bmatrix}$$
$$=(2-\lambda)(-1-\lambda)(4-\lambda)=-\lambda^3+5\lambda^2-2\lambda-8$$

13. Make a cofactor expansion down the third column:

$$\det(A-\lambda I)=\det\begin{bmatrix}6-\lambda & -2 & 0\\ -2 & 9-\lambda & 0\\ 5 & 8 & 3-\lambda\end{bmatrix}=(3-\lambda)\cdot\det\begin{bmatrix}6-\lambda & -2\\ -2 & 9-\lambda\end{bmatrix}$$
$$=(3-\lambda)[(6-\lambda)(9-\lambda)-(-2)(-2)]=(3-\lambda)(\lambda^2-15\lambda+50)$$
$$=-\lambda^3+18\lambda^2-95\lambda+150 \text{ or } (3-\lambda)(\lambda-5)(\lambda-10)$$

14. Make a cofactor expansion along the second row:

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda & -2 & 3\\ 0 & 1-\lambda & 0\\ 6 & 7 & -2-\lambda\end{bmatrix}=(1-\lambda)\cdot\det\begin{bmatrix}5-\lambda & 3\\ 6 & -2-\lambda\end{bmatrix}$$
$$=(1-\lambda)\cdot[(5-\lambda)(-2-\lambda)-3\cdot6]=(1-\lambda)(\lambda^2-3\lambda-28)$$
$$=-\lambda^3+4\lambda^2+25\lambda-28 \text{ or } (1-\lambda)(\lambda-7)(\lambda+4)$$

15. Use the fact that the determinant of a triangular matrix is the product of the diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & -7 & 0 & 2\\ 0 & 3-\lambda & -4 & 6\\ 0 & 0 & 3-\lambda & -8\\ 0 & 0 & 0 & 1-\lambda\end{bmatrix}=(4-\lambda)(3-\lambda)^2(1-\lambda)$$

The eigenvalues are 4, 3, 3, and 1.

ffective.

olynomial

16. The determinant of a triangular matrix is the product of its diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda & 0 & 0 & 0\\ 8 & -4-\lambda & 0 & 0\\ 0 & 7 & 1-\lambda & 0\\ 1 & -5 & 2 & 1-\lambda\end{bmatrix}=(5-\lambda)(-4-\lambda)(1-\lambda)^2$$

The eigenvalues are 5, 1, 1, and −4.

17. The determinant of a triangular matrix is the product of its diagonal entries:

$$\begin{bmatrix}3-\lambda & 0 & 0 & 0 & 0\\ -5 & 1-\lambda & 0 & 0 & 0\\ 3 & 8 & 0-\lambda & 0 & 0\\ 0 & -7 & 2 & 1-\lambda & 0\\ -4 & 1 & 9 & -2 & 3-\lambda\end{bmatrix}=(3-\lambda)^2(1-\lambda)^2(-\lambda)$$

The eigenvalues are 3, 3, 1, 1, and 0.

18. Row reduce the augmented matrix for the equation $(A-5I)\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}0 & -2 & 6 & -1 & 0\\ 0 & -2 & h & 0 & 0\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & -4 & 0\end{bmatrix}\sim\begin{bmatrix}0 & -2 & 6 & -1 & 0\\ 0 & 0 & h-6 & 1 & 0\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 4 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 1 & -3 & 0 & 0\\ 0 & 0 & h-6 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

For a two-dimensional eigenspace, the system above needs two free variables. This happens if and only if $h=6$.

19. Since the equation $\det(A-\lambda I)=(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$ holds for all λ, set $\lambda=0$ and conclude that $\det A=\lambda_1\lambda_2\cdots\lambda_n$.

20. $\det(A^T-\lambda I)=\det(A^T-\lambda I^T)$

$=\det(A-\lambda I)^T$ Transpose property

$=\det(A-\lambda I)$ Theorem 3(c)

21. **a.** False. See Example 1.

b. False. See Theorem 3.

c. True. See Theorem 3.

d. False. See the solution of Example 4.

22. **a.** False. See the paragraph before Theorem 3.

b. False. See Theorem 3.

c. True. See the paragraph before Example 4.

d. False. See the warning after Theorem 4.

23. If $A=QR$, with Q invertible, and if $A_1=RQ$, then write $A_1=Q^{-1}QRQ=Q^{-1}AQ$, which shows that A_1 is similar to A.

5. a. $T(\mathbf{p})=(t+5)(2-t+t^2)=10-3t+4t^2+t^3$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$\begin{aligned}T(\mathbf{p}(t)+\mathbf{q}(t))&=(t+5)[\mathbf{p}(t)+\mathbf{q}(t)]=(t+5)\mathbf{p}(t)+(t+5)\mathbf{q}(t)\\&=T(\mathbf{p}(t))+T(\mathbf{q}(t))\end{aligned}$$

$$\begin{aligned}T(c\cdot\mathbf{p}(t))&=(t+5)[c\cdot\mathbf{p}(t)]=c\cdot(t+5)\mathbf{p}(t)\\&=c\cdot T[\mathbf{p}(t)]\end{aligned}$$

and T is a linear transformation.

c. Let $B=\{1,t,t^2\}$ and $C=\{1,t,t^2,t^3\}$. Since $T(\mathbf{b}_1)=T(1)=(t+5)(1)=t+5$, $[T(\mathbf{b}_1)]_C=\begin{bmatrix}5\\1\\0\\0\end{bmatrix}$. Li

since $T(\mathbf{b}_2)=T(t)=(t+5)(t)=t^2+5t$, $[T(\mathbf{b}_2)]_C=\begin{bmatrix}0\\5\\1\\0\end{bmatrix}$, and since

$T(\mathbf{b}_3)=T(t^2)=(t+5)(t^2)=t^3+5t^2$, $[T(\mathbf{b}_3)]_C=\begin{bmatrix}0\\0\\5\\1\end{bmatrix}$. Thus the matrix for T relative to B and

C is $[\,[T(\mathbf{b}_1)]_C\ \ [T(\mathbf{b}_2)]_C\ \ [T(\mathbf{b}_3)]_C\,]=\begin{bmatrix}5&0&0\\1&5&0\\0&1&5\\0&0&1\end{bmatrix}$.

6. a. $T(\mathbf{p})=(2-t+t^2)+t^2(2-t+t^2)=2-t+3t^2-t^3+t^4$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$\begin{aligned}T(\mathbf{p}(t)+\mathbf{q}(t))&=[\mathbf{p}(t)+\mathbf{q}(t)]+t^2[\mathbf{p}(t)+\mathbf{q}(t)]\\&=[\mathbf{p}(t)+t^2\mathbf{p}(t)]+[\mathbf{q}(t)+t^2\mathbf{q}(t)]\\&=T(\mathbf{p}(t))+T(\mathbf{q}(t))\end{aligned}$$

$$\begin{aligned}T(c\cdot\mathbf{p}(t))&=[c\cdot\mathbf{p}(t)]+t^2[c\cdot\mathbf{p}(t)]\\&=c\cdot[\mathbf{p}(t)+t^2\mathbf{p}(t)]\\&=c\cdot T[\mathbf{p}(t)]\end{aligned}$$

and T is a linear transformation.

c. Let $B=\{1,t,t^2\}$ and $C=\{1,t,t^2,t^3,t^4\}$. Since $T(\mathbf{b}_1)=T(1)=1+t^2(1)=t^2+1$, $[T(\mathbf{b}_1)]_C=\begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}$.

Likewise since $T(\mathbf{b}_2)=T(t)=t+(t^2)(t)=t^3+t$, $[T(\mathbf{b}_2)]_C=\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}$, and

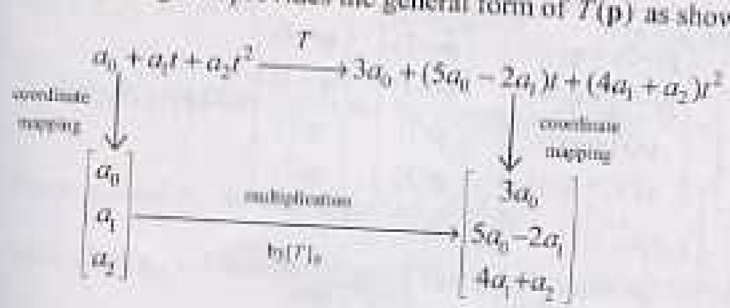
since $T(\mathbf{b}_3)=T(t^2)=t^2+(t^2)(t^2)=t^4+t^2$, $[T(\mathbf{b}_3)]_C=\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}$. Thus the matrix for T relative to

B and C is $[\,[T(\mathbf{b}_1)]_C \quad [T(\mathbf{b}_2)]_C \quad [T(\mathbf{b}_3)]_C\,]=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

7. Since $T(\mathbf{b}_1)=T(1)=3+5t$, $[T(\mathbf{b}_1)]_B=\begin{bmatrix}3\\5\\0\end{bmatrix}$. Likewise since $T(\mathbf{b}_2)=T(t)=-2t+4t^2$, $[T(\mathbf{b}_2)]_B=\begin{bmatrix}0\\-2\\4\end{bmatrix}$,

and since $T(\mathbf{b}_3)=T(t^2)=t^2$, $[T(\mathbf{b}_3)]_B=\begin{bmatrix}0\\0\\1\end{bmatrix}$. Thus the matrix representation of T relative to the basis

B is $[\,[T(\mathbf{b}_1)]_B \quad [T(\mathbf{b}_2)]_B \quad [T(\mathbf{b}_3)]_B\,]=\begin{bmatrix}3&0&0\\5&-2&0\\0&4&1\end{bmatrix}$. Perhaps a faster way is to realize that the information given provides the general form of $T(\mathbf{p})$ as shown in the figure below:



The matrix that implements the multiplication along the bottom of the figure is easily filled in by inspection:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3a_0 \\ 5a_0 - 2a_1 \\ 4a_1 + a_2 \end{bmatrix} \text{ implies that } [T]_B = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

8. Since $[3\mathbf{b}_1 - 4\mathbf{b}_2]_B = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$, $[T(3\mathbf{b}_1 - 4\mathbf{b}_2)]_B = [T]_B[3\mathbf{b}_1 - 4\mathbf{b}_2]_B = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$

and $T(3\mathbf{b}_1 - 4\mathbf{b}_2) = 24\mathbf{b}_1 - 20\mathbf{b}_2 + 11\mathbf{b}_3$.

9. a. $T(\mathbf{p}) = \begin{bmatrix} 5+3(-1) \\ 5+3(0) \\ 5+3(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

b. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(0) \\ (\mathbf{p}+\mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1)+\mathbf{q}(-1) \\ \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}(1)+\mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-1) \\ \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-1) \\ (c\cdot\mathbf{p})(0) \\ (c\cdot\mathbf{p})(1) \end{bmatrix} = \begin{bmatrix} c\cdot(\mathbf{p}(-1)) \\ c\cdot(\mathbf{p}(0)) \\ c\cdot(\mathbf{p}(1)) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = c\cdot T(\mathbf{p})$$

and T is a linear transformation.

c. Let $B = \{1, t, t^2\}$ and $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Since

$[T(\mathbf{b}_1)]_{\mathcal{E}} = T(\mathbf{b}_1) = T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $[T(\mathbf{b}_2)]_{\mathcal{E}} = T(\mathbf{b}_2) = T(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $[T(\mathbf{b}_3)]_{\mathcal{E}} = T(\mathbf{b}_3) = T(t^2) =$

the matrix for T relative to B and $\mathcal{E}$ is $\begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{E}} & [T(\mathbf{b}_2)]_{\mathcal{E}} & [T(\mathbf{b}_3)]_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

10. a. Let $\mathbf{p}$ and $\mathbf{q}$ be polynomials in $\mathbb{P}_3$, and let c be any scalar. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-3) \\ (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(1) \\ (\mathbf{p}+\mathbf{q})(3) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-3)+\mathbf{q}(-3) \\ \mathbf{p}(-1)+\mathbf{q}(-1) \\ \mathbf{p}(1)+\mathbf{q}(1) \\ \mathbf{p}(3)+\mathbf{q}(3) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-3) \\ \mathbf{q}(-1) \\ \mathbf{q}(1) \\ \mathbf{q}(3) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-3) \\ (c\cdot\mathbf{p})(-1) \\ (c\cdot\mathbf{p})(1) \\ (c\cdot\mathbf{p})(3) \end{bmatrix} = \begin{bmatrix} c\cdot(\mathbf{p}(-3)) \\ c\cdot(\mathbf{p}(-1)) \\ c\cdot(\mathbf{p}(1)) \\ c\cdot(\mathbf{p}(3)) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-3) \\ \mathbf{p}(-1) \\ \mathbf{p}(1) \\ \mathbf{p}(3) \end{bmatrix} = c\cdot T(\mathbf{p})$$

and T is a linear transformation.