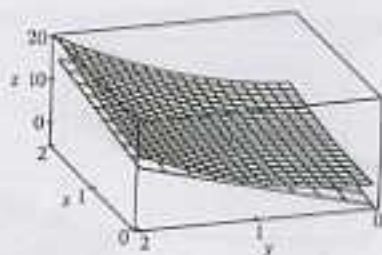
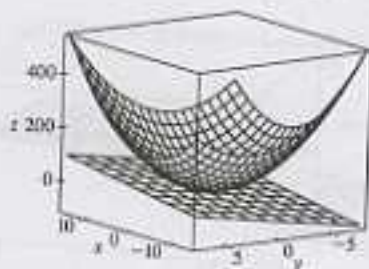


## 11.4

### Tangent Planes and Linear Approximations . . . . .

1.  $z = f(x, y) = 4x^2 - y^2 + 2y \Rightarrow f_x(x, y) = 8x, f_y(x, y) = -2y + 2$ , so  $f_x(-1, 2) = -8, f_y(-1, 2) = -2$ .  
By Equation 2, an equation of the tangent plane is  $z - 4 = f_x(-1, 2)[x - (-1)] + f_y(-1, 2)(y - 2) \Rightarrow$   
 $z - 4 = -8(x + 1) - 2(y - 2)$  or  $z = -8x - 2y$ .
2.  $z = f(x, y) = e^{x^2 - y^2} \Rightarrow f_x(x, y) = 2xe^{x^2 - y^2}, f_y(x, y) = -2ye^{x^2 - y^2}$ , so  $f_x(1, -1) = 2, f_y(1, -1) = 2$ .  
By Equation 2, an equation of the tangent plane is  $z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)(y - (-1)) \Rightarrow$   
 $z - 1 = 2(x - 1) + 2(y + 1)$  or  $z = 2x + 2y + 1$ .
3.  $z = f(x, y) = \sqrt{4 - x^2 - 2y^2} \Rightarrow f_x(x, y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{4 - x^2 - 2y^2}}$ ,  
 $f_y(x, y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-4y) = -\frac{2y}{\sqrt{4 - x^2 - 2y^2}}$ , so  $f_x(1, -1) = -1$  and  $f_y(1, -1) = 2$ . Thus, an  
equation of the tangent plane is  $z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)(y - (-1)) \Rightarrow$   
 $z - 1 = -1(x - 1) + 2(y + 1)$  or  $x - 2y + z = 4$ .
4.  $z = f(x, y) = y \ln x \Rightarrow f_x(x, y) = y/x, f_y(x, y) = \ln x$ , so  $f_x(1, 4) = 4, f_y(1, 4) = 0$ , and an equation of  
the tangent plane is  $z - 0 = f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4) \Rightarrow z = 4(x - 1) + 0(y - 4)$  or  $z = 4x - 4$ .
5.  $z = f(x, y) = x^2 + xy + 3y^2$ , so  $f_x(x, y) = 2x + y \Rightarrow f_x(1, 1) = 3, f_y(x, y) = x + 6y \Rightarrow f_y(1, 1) = 7$   
and an equation of the tangent plane is  $z - 5 = 3(x - 1) + 7(y - 1)$  or  $z = 3x + 7y - 5$ . After zooming in, the  
surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we  
zoom in farther, the surface and the tangent plane will appear to coincide.



6.  $z = f(x, y) = \sqrt{x - y}$ , so  $f_x(x, y) = \frac{1}{2}(x - y)^{-1/2}, f_x(5, 1) = \frac{1}{4}, f_y(x, y) = -\frac{1}{2}(x - y)^{-1/2}, f_y(5, 1) = -\frac{1}{4}$ ,  
and an equation of the tangent plane is  $z - 2 = \frac{1}{4}(x - 5) - \frac{1}{4}(y - 1)$  or  $z = \frac{1}{4}x - \frac{1}{4}y + 1$ . After zooming in, the  
surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we  
zoom in farther, the surface and the tangent plane will appear to coincide.



11.  $f(x, y) = \tan^{-1}(x + 2y)$ . The partial derivatives are  $f_x(x, y) = \frac{1}{1 + (x + 2y)^2}$  and  $f_y(x, y) = \frac{2}{1 + (x + 2y)^2}$ , so  $f_x(1, 0) = \frac{1}{2}$  and  $f_y(1, 0) = 1$ . Both  $f_x$  and  $f_y$  are continuous functions, so  $f$  is differentiable at  $(1, 0)$ , and the linearization of  $f$  at  $(1, 0)$  is  $L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) = \frac{\pi}{4} + \frac{1}{2}(x - 1) + 1(y) = \frac{1}{2}x + y + \frac{\pi}{4} - \frac{1}{2}$ .
12.  $f(x, y) = \sin(2x + 3y)$ . The partial derivatives are  $f_x(x, y) = 2 \cos(2x + 3y)$  and  $f_y(x, y) = 3 \cos(2x + 3y)$ , so  $f_x(-3, 2) = 2$  and  $f_y(-3, 2) = 3$ . Both  $f_x$  and  $f_y$  are continuous functions, so  $f$  is differentiable at  $(-3, 2)$ , and the linearization of  $f$  at  $(-3, 2)$  is  $L(x, y) = f(-3, 2) + f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) = 0 + 2(x + 3) + 3(y - 2) = 2x + 3y$ .
13.  $f(x, y) = \sqrt{20 - x^2 - 7y^2} \Rightarrow f_x(x, y) = -\frac{x}{\sqrt{20 - x^2 - 7y^2}}$  and  $f_y(x, y) = -\frac{7y}{\sqrt{20 - x^2 - 7y^2}}$ , so  $f_x(2, 1) = -\frac{2}{3}$  and  $f_y(2, 1) = -\frac{7}{3}$ . Then the linear approximation of  $f$  at  $(2, 1)$  is given by

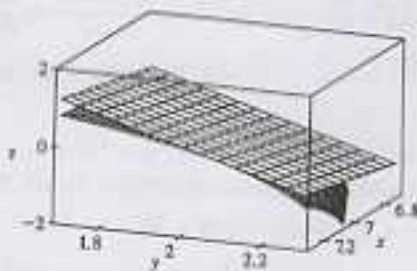
$$f(x, y) \approx f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) = 3 - \frac{2}{3}(x - 2) - \frac{7}{3}(y - 1) \\ = -\frac{2}{3}x - \frac{7}{3}y + \frac{20}{3}$$

$$\text{Thus } f(1.95, 1.08) \approx -\frac{2}{3}(1.95) - \frac{7}{3}(1.08) + \frac{20}{3} = 2.84\bar{6}.$$

14.  $f(x, y) = \ln(x - 3y) \Rightarrow f_x(x, y) = \frac{1}{x - 3y}$  and  $f_y(x, y) = -\frac{3}{x - 3y}$ , so  $f_x(7, 2) = 1$  and  $f_y(7, 2) = -3$ . Then the linear approximation of  $f$  at  $(7, 2)$  is given by

$$f(x, y) \approx f(7, 2) + f_x(7, 2)(x - 7) + f_y(7, 2)(y - 2) \\ = 0 + 1(x - 7) - 3(y - 2) = x - 3y - 1$$

Thus  $f(6.9, 2.06) \approx 6.9 - 3(2.06) - 1 = -0.28$ . The graph shows that our approximated value is slightly greater than the actual value.



15.  $f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \Rightarrow f_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$ ,  $f_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$ , and  $f_z(x, y, z) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$ , so  $f_x(3, 2, 6) = \frac{3}{7}$ ,  $f_y(3, 2, 6) = \frac{2}{7}$ , and  $f_z(3, 2, 6) = \frac{6}{7}$ . Then the linear approximation of  $f$  at  $(3, 2, 6)$  is given by

$$f(x, y, z) \approx f(3, 2, 6) + f_x(3, 2, 6)(x - 3) + f_y(3, 2, 6)(y - 2) + f_z(3, 2, 6)(z - 6) \\ = 7 + \frac{3}{7}(x - 3) + \frac{2}{7}(y - 2) + \frac{6}{7}(z - 6) = \frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z$$

$$\text{Thus } \sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914.$$

Averaging these values gives  $f_T(16, 30) \approx 1.375$ . Similarly,  $f_v(16, 30) = \lim_{h \rightarrow 0} \frac{f(16, 30+h) - f(16, 30)}{h}$ ,

$$\text{so we use } h = \pm 10: f_v(16, 30) \approx \frac{f(16, 40) - f(16, 30)}{10} = \frac{7 - 9}{10} = -0.2,$$

$f_v(16, 30) \approx \frac{f(16, 20) - f(16, 30)}{-10} = \frac{11 - 9}{-10} = -0.2$ . Averaging these values gives  $f_v(16, 30) \approx -0.2$ . The linear approximation, then, is

$$\begin{aligned} f(T, v) &\approx f(16, 30) + f_T(16, 30)(T - 16) + f_v(16, 30)(v - 30) \\ &\approx 9 + 1.375(T - 16) - 0.2(v - 30) \end{aligned}$$

Thus when  $T = 14$  and  $v = 27$ ,  $f(14, 27) \approx 9 + 1.375(14 - 16) - 0.2(27 - 30) = 6.85$ , so we estimate the wind-chill index to be approximately  $6.85^\circ\text{C}$ .

$$19. u = e^t \sin \theta \Rightarrow du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial \theta} d\theta = e^t \sin \theta dt + e^t \cos \theta d\theta$$

$$20. v = y \cos xy \Rightarrow$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = y(-\sin xy)y dx + [y(-\sin xy)x + \cos xy] dy = -y^2 \sin xy dx + (\cos xy - xy \sin xy) dy$$

$$21. w = \ln \sqrt{x^2 + y^2 + z^2} \Rightarrow$$

$$\begin{aligned} dw &= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz \\ &= \left( \frac{1}{2} \right) \frac{2x(x^2 + y^2 + z^2)^{-1/2} dx + 2y(x^2 + y^2 + z^2)^{-1/2} dy + 2z(x^2 + y^2 + z^2)^{-1/2} dz}{(x^2 + y^2 + z^2)^{1/2}} \\ &= \frac{x dx + y dy + z dz}{x^2 + y^2 + z^2} \end{aligned}$$

$$22. u = \frac{r}{s + 2t} \Rightarrow$$

$$\begin{aligned} du &= \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt = \frac{1}{s + 2t} dr + r(-1)(s + 2t)^{-2} ds + r(-1)(s + 2t)^{-2}(2) dt \\ &= \frac{1}{s + 2t} dr - \frac{r}{(s + 2t)^2} ds - \frac{2r}{(s + 2t)^2} dt \end{aligned}$$

$$23. dx = \Delta x = 0.05, dy = \Delta y = 0.1, z = 5x^2 + y^2, z_x = 10x, z_y = 2y. \text{ Thus when } x = 1 \text{ and } y = 2, dz = z_x(1, 2) dx + z_y(1, 2) dy = (10)(0.05) + (4)(0.1) = 0.9 \text{ while } \Delta z = f(1.05, 2.1) - f(1, 2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225.$$

$$24. dx = \Delta x = -0.04, dy = \Delta y = 0.05, z = x^2 - xy + 3y^2, z_x = 2x - y, z_y = 6y - x. \text{ Thus when } x = 3 \text{ and } y = -1, dz = (7)(-0.04) + (-9)(0.05) = -0.73 \text{ while } \Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9 + 3 + 3) = -0.7189.$$

$$25. dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy = y dx + x dy \text{ and } |\Delta x| \leq 0.1, |\Delta y| \leq 0.1. \text{ We use } dx = 0.1, dy = 0.1 \text{ with } x = 30, y = 24; \text{ then the maximum error in the area is about } dA = 24(0.1) + 30(0.1) = 5.4 \text{ cm}^2.$$

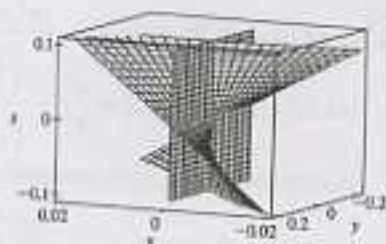
$$26. \text{ Let } S \text{ be surface area. Then } S = 2(xy + xz + yz) \text{ and } dS = 2(y + z) dx + 2(x + z) dy + 2(x + y) dz. \text{ The maximum error occurs with } \Delta x = \Delta y = \Delta z = 0.2. \text{ Using } dx = \Delta x, dy = \Delta y, dz = \Delta z \text{ we find the maximum error in calculated surface area to be about } dS = (220)(0.2) + (260)(0.2) + (280)(0.2) = 152 \text{ cm}^2.$$

$$35. \mathbf{r}(u, v) = uv\mathbf{i} + ue^v\mathbf{j} + ve^u\mathbf{k}$$

$$\mathbf{r}_u = \langle v, e^v, ve^u \rangle, \mathbf{r}_v = \langle u, ue^u, e^u \rangle, \text{ and}$$

$$\mathbf{r}_u \times \mathbf{r}_v = e^{u+v}(1-uv)\mathbf{i} + e^u S(uv-v)\mathbf{j} + e^v(uv-u)\mathbf{k}.$$

The point  $(0, 0, 0)$  corresponds to  $u = 0, v = 0$ . Thus a normal vector to the surface at  $(0, 0, 0)$  is  $\mathbf{i}$ , and an equation of the tangent plane is  $x = 0$ .

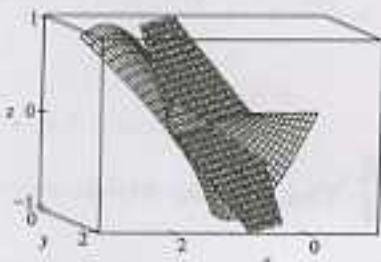


$$36. \mathbf{r}(u, v) = (u+v)\mathbf{i} + u\cos v\mathbf{j} + v\sin u\mathbf{k}$$

$$\mathbf{r}_u = \langle 1, \cos v, v\cos u \rangle, \mathbf{r}_v = \langle 1, -u\sin v, \sin u \rangle, \text{ and}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v \sin u + uv \cos u \sin v, \\ v \cos u - \sin u, -u \sin v - \cos v \rangle$$

The point  $(1, 1, 0)$  corresponds to  $u = 1, v = 0$ . Thus a normal vector to the surface at  $(1, 1, 0)$  is  $\langle \sin 1, -\sin 1, -1 \rangle$ , and an equation of the tangent plane is  $(\sin 1)x - (\sin 1)y - z = 0$ .



$$37. \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)^2 + (b + \Delta y)^2 - (a^2 + b^2)$$

$$= a^2 + 2a\Delta x + (\Delta x)^2 + b^2 + 2b\Delta y + (\Delta y)^2 - a^2 - b^2 = 2a\Delta x + (\Delta x)^2 + 2b\Delta y + (\Delta y)^2$$

But  $f_x(a, b) = 2a$  and  $f_y(a, b) = 2b$  and so  $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \Delta x\Delta x + \Delta y\Delta y$ , which is Definition 7 with  $\epsilon_1 = \Delta x$  and  $\epsilon_2 = \Delta y$ . Hence  $f$  is differentiable.

$$38. \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) = (a + \Delta x)(b + \Delta y) - 5(b + \Delta y)^2 - (ab - 5b^2)$$

$$= ab + a\Delta y + b\Delta x + \Delta x\Delta y - 5b^2 - 10b\Delta y - 5(\Delta y)^2 - ab + 5b^2$$

$$= (a - 10b)\Delta y + b\Delta x + \Delta x\Delta y - 5\Delta y\Delta y,$$

but  $f_x(a, b) = b$  and  $f_y(a, b) = a - 10b$  and so  $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \Delta x\Delta y - 5\Delta y\Delta y$ , which is Definition 7 with  $\epsilon_1 = \Delta y$  and  $\epsilon_2 = -5\Delta y$ . Hence  $f$  is differentiable.

39. To show that  $f$  is continuous at  $(a, b)$  we need to show that  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$  or equivalently

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b). \text{ Since } f \text{ is differentiable at } (a, b),$$

$f(a + \Delta x, b + \Delta y) - f(a, b) = \Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$ , where  $\epsilon_1$  and  $\epsilon_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ . Thus  $f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$ . Taking the limit of both sides as  $(\Delta x, \Delta y) \rightarrow (0, 0)$  gives  $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} f(a + \Delta x, b + \Delta y) = f(a, b)$ . Thus  $f$  is continuous at  $(a, b)$ .

$$40. (a) \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0 \text{ and } \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0. \text{ Thus}$$

$f_x(0, 0) = f_y(0, 0) = 0$ . To show that  $f$  isn't differentiable at  $(0, 0)$  we need only show that  $f$  is not continuous at  $(0, 0)$  and apply Exercise 39. As  $(x, y) \rightarrow (0, 0)$  along the  $x$ -axis  $f(x, y) = 0/x^2 = 0$  for  $x \neq 0$  so

$f(x, y) \rightarrow 0$  as  $(x, y) \rightarrow (0, 0)$  along the  $x$ -axis. But as  $(x, y) \rightarrow (0, 0)$  along the line  $y = x$ ,

$f(x, x) = x^2/(2x^2) = 1/2$  for  $x \neq 0$  so  $f(x, y) \rightarrow 1/2$  as  $(x, y) \rightarrow (0, 0)$  along this line. Thus  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$

doesn't exist, so  $f$  is discontinuous at  $(0, 0)$  and thus not differentiable there.

- (b) For  $(x, y) \neq (0, 0)$ ,  $f_x(x, y) = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$ . If we approach  $(0, 0)$  along the  $y$ -axis, then  $f_x(x, y) = f_x(0, y) = \frac{y^3}{y^4} = \frac{1}{y}$ , so  $f_x(x, y) \rightarrow \pm\infty$  as  $(x, y) \rightarrow (0, 0)$ . Thus  $\lim_{(x, y) \rightarrow (0, 0)} f_x(x, y)$  does not exist and  $f_x(x, y)$  is not continuous at  $(0, 0)$ . Similarly,  $f_y(x, y) = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$  for  $(x, y) \neq (0, 0)$ , and if we approach  $(0, 0)$  along the  $x$ -axis, then  $f_y(x, y) = f_y(x, 0) = \frac{x^3}{x^4} = \frac{1}{x}$ . Thus  $\lim_{(x, y) \rightarrow (0, 0)} f_y(x, y)$  does not exist and  $f_y(x, y)$  is not continuous at  $(0, 0)$ .

## 11.5

### The Chain Rule

- $z = \sin x \cos y$ ,  $x = \pi t$ ,  $y = \sqrt{t}$   $\Rightarrow$   

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \cos x \cos y \cdot \pi + \sin x (-\sin y) \cdot \frac{1}{2} t^{-1/2} = \pi \cos x \cos y - \frac{1}{2\sqrt{t}} \sin x \sin y$$
- $z = x \ln(x + 2y)$ ,  $x = \sin t$ ,  $y = \cos t$   $\Rightarrow$   

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \left[ x \cdot \frac{1}{x+2y} + 1 \cdot \ln(x+2y) \right] \cos t + x \cdot \frac{1}{x+2y} (2) \cdot (-\sin t) \\ &= \left[ \frac{x}{x+2y} + \ln(x+2y) \right] \cos t - \frac{2x}{x+2y} (\sin t) \end{aligned}$$
- $w = xe^{y/z}$ ,  $x = t^2$ ,  $y = 1 - t$ ,  $z = 1 + 2t$   $\Rightarrow$   

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = e^{y/z} \cdot 2t + xe^{y/z} \left( \frac{1}{z} \right) \cdot (-1) + xe^{y/z} \left( -\frac{y}{z^2} \right) \cdot 2 = e^{y/z} \left( 2t - \frac{x}{z} - \frac{2xy}{z^2} \right)$$
- $w = xy + yz^2$ ,  $x = e^t$ ,  $y = e^t \sin t$ ,  $z = e^t \cos t$   $\Rightarrow$   

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = y \cdot e^t + (x + z^2) \cdot (e^t \cos t + e^t \sin t) + 2yz \cdot (-e^t \sin t + e^t \cos t) \\ &= e^t [y + (x + z^2)(\cos t + \sin t) + 2yz(\cos t - \sin t)] \end{aligned}$$
- $z = x^2 + xy + y^2$ ,  $x = s + t$ ,  $y = st$   $\Rightarrow$   

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(t) = 2x + y + xt + 2yt \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (2x + y)(1) + (x + 2y)(s) = 2x + y + xs + 2ys \end{aligned}$$
- $z = \frac{x}{y}$ ,  $x = se^t$ ,  $y = 1 + se^{-t}$   $\Rightarrow$   

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \frac{1}{y} (e^t) + \left( -\frac{x}{y^2} \right) (e^{-t}) = \frac{1}{y} e^t - \frac{x}{y^2} e^{-t} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \frac{1}{y} (se^t) + \left( -\frac{x}{y^2} \right) (-se^{-t}) = \frac{s}{y} e^t + \frac{xs}{y^2} e^{-t} \end{aligned}$$

$$7. z = e^r \cos \theta, r = st, \theta = \sqrt{s^2 + t^2} \Rightarrow$$

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial s} = e^r \cos \theta \cdot t + e^r (-\sin \theta) \cdot \frac{1}{2} (s^2 + t^2)^{-1/2} (2s) \\ &= te^r \cos \theta - e^r \sin \theta \cdot \frac{s}{\sqrt{s^2 + t^2}} = e^r \left( t \cos \theta - \frac{s}{\sqrt{s^2 + t^2}} \sin \theta \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial t} = e^r \cos \theta \cdot s + e^r (-\sin \theta) \cdot \frac{1}{2} (s^2 + t^2)^{-1/2} (2t) \\ &= se^r \cos \theta - e^r \sin \theta \cdot \frac{t}{\sqrt{s^2 + t^2}} = e^r \left( s \cos \theta - \frac{t}{\sqrt{s^2 + t^2}} \sin \theta \right) \end{aligned}$$

$$8. z = \sin \alpha \tan \beta, \alpha = 3s + t, \beta = s - t \Rightarrow$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial s} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial s} = \cos \alpha \tan \beta \cdot 3 + \sin \alpha \sec^2 \beta \cdot 1 = 3 \cos \alpha \tan \beta + \sin \alpha \sec^2 \beta$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial t} = \cos \alpha \tan \beta \cdot 1 + \sin \alpha \sec^2 \beta \cdot (-1) = \cos \alpha \tan \beta - \sin \alpha \sec^2 \beta$$

9. When  $t = 3$ ,  $x = g(3) = 2$  and  $y = h(3) = 7$ . By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = f_x(2, 7)g'(3) + f_y(2, 7)h'(3) = (6)(5) + (-8)(-4) = 62.$$

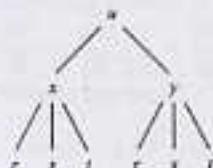
10. By the Chain Rule (3),  $\frac{\partial W}{\partial s} = \frac{\partial W}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial W}{\partial v} \frac{\partial v}{\partial s}$ . Then

$$\begin{aligned} W_s(1, 0) &= F_u(u(1, 0), v(1, 0))u_s(1, 0) + F_v(u(1, 0), v(1, 0))v_s(1, 0) \\ &= F_u(2, 3)u_s(1, 0) + F_v(2, 3)v_s(1, 0) = (-1)(-2) + (10)(5) = 52 \end{aligned}$$

Similarly,  $\frac{\partial W}{\partial t} = \frac{\partial W}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial W}{\partial v} \frac{\partial v}{\partial t} \Rightarrow$

$$\begin{aligned} W_t(1, 0) &= F_u(u(1, 0), v(1, 0))u_t(1, 0) + F_v(u(1, 0), v(1, 0))v_t(1, 0) \\ &= F_u(2, 3)u_t(1, 0) + F_v(2, 3)v_t(1, 0) = (-1)(6) + (10)(4) = 34 \end{aligned}$$

11.



$$u = f(x, y), x = x(r, s, t), y = y(r, s, t) \Rightarrow$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}, \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

12.



$$w = f(x, y, z), x = x(t, u), y = y(t, u), z = z(t, u) \Rightarrow$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$$

21.  $\cos(x - y) = xe^y$ , so let  $F(x, y) = \cos(x - y) - xe^y = 0$ .

$$\text{Then } \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(x - y) - e^y}{-\sin(x - y)(-1) - xe^y} = \frac{\sin(x - y) + e^y}{\sin(x - y) - xe^y}.$$

22.  $\sin x + \cos y = \sin x \cos y$ , so let  $F(x, y) = \sin x + \cos y - \sin x \cos y = 0$ . Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\cos x - \cos x \cos y}{-\sin y + \sin x \sin y} = \frac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}.$$

23.  $xy^2 + yz^2 + xz^2 = 3$ , so let  $F(x, y, z) = xy^2 + yz^2 + xz^2 - 3 = 0$ .

$$\text{Then } \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y^2 + 2xz}{2yz + x^2} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2xy + z^2}{2yz + x^2}.$$

24.  $xyz = \cos(x + y + z)$ . Let  $F(x, y, z) = xyz - \cos(x + y + z) = 0$ , so

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x + y + z)}{xy + \sin(x + y + z)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x + y + z)}{xy + \sin(x + y + z)}.$$

25. Let  $F(x, y, z) = xe^y + yz + ze^x = 0$ . Then  $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^y + ze^x}{y + e^x}$ ,  $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xe^y + z}{y + e^x}$ .

26.  $\ln(x + yz) = 1 + xy^2z^3$ , so let  $F(x, y, z) = \ln(x + yz) - 1 - xy^2z^3 = 0$ . Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1/(x + yz) - y^2z^3}{y/(x + yz) - 3xy^2z^2} = \frac{y^2z^3(x + yz) - 1}{y - 3xy^2z^2(x + yz)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z/(x + yz) - 2xyz^3}{y/(x + yz) - 3xy^2z^2} = \frac{2xyz^3(x + yz) - z}{y - 3xy^2z^2(x + yz)}$$

27. Since  $x$  and  $y$  are each functions of  $t$ ,  $T(x, y)$  is a function of  $t$ , so by the Chain Rule,  $\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt}$ .

After 3 seconds,  $x = \sqrt{1+t} = \sqrt{1+3} = 2$ ,  $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$ ,  $\frac{dx}{dt} = \frac{1}{2\sqrt{1+t}} = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$ ,

and  $\frac{dy}{dt} = \frac{1}{3}$ . Then  $\frac{dT}{dt} = T_x(2, 3) \frac{dx}{dt} + T_y(2, 3) \frac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$ . Thus the temperature is rising at a rate of  $2^\circ\text{C/s}$ .

28. (a) Since  $\partial W/\partial T$  is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since  $\partial W/\partial R$  is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of  $0.15^\circ\text{C/year}$ , we know that  $dT/dt = 0.15$ . Since rainfall is decreasing at a rate of  $0.1 \text{ cm/year}$ , we know  $dR/dt = -0.1$ . Then, by the Chain Rule,

$$\frac{dW}{dt} = \frac{\partial W}{\partial T} \frac{dT}{dt} + \frac{\partial W}{\partial R} \frac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1. \text{ Thus we estimate that wheat production will decrease at a rate of } 1.1 \text{ units/year.}$$

29.  $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$ , so  $\frac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$  and

$$\frac{\partial C}{\partial D} = 0.016. \text{ According to the graph, the diver is experiencing a temperature of approximately } 12.5^\circ\text{C at}$$

$$t = 20 \text{ minutes, so } \frac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36. \text{ By sketching tangent lines at } t = 20 \text{ to the}$$

graphs given, we estimate  $\frac{dD}{dt} \approx \frac{1}{2}$  and  $\frac{dT}{dt} \approx -\frac{1}{10}$ . Then, by the Chain Rule,

$$\frac{dC}{dt} = \frac{\partial C}{\partial T} \frac{dT}{dt} + \frac{\partial C}{\partial D} \frac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33. \text{ Thus the speed of sound experienced by the}$$

$$30. V = \pi r^2 h/3, \text{ so } \frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = \frac{2\pi r h}{3} 1.8 + \frac{\pi r^2}{3} (-2.5) = 20,160\pi - 12,000\pi = 8160\pi \text{ in}^3/\text{s}.$$

31. (a)  $V = \ell wh$ , so by the Chain Rule,

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial \ell} \frac{d\ell}{dt} + \frac{\partial V}{\partial w} \frac{dw}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = wh \frac{d\ell}{dt} + \ell h \frac{dw}{dt} + \ell w \frac{dh}{dt} \\ &= 2 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot (-3) = 6 \text{ m}^3/\text{s} \end{aligned}$$

(b)  $S = 2(\ell w + \ell h + wh)$ , so by the Chain Rule,

$$\begin{aligned} \frac{dS}{dt} &= \frac{\partial S}{\partial \ell} \frac{d\ell}{dt} + \frac{\partial S}{\partial w} \frac{dw}{dt} + \frac{\partial S}{\partial h} \frac{dh}{dt} = 2(w+h) \frac{d\ell}{dt} + 2(\ell+h) \frac{dw}{dt} + 2(\ell+w) \frac{dh}{dt} \\ &= 2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10 \text{ m}^2/\text{s} \end{aligned}$$

$$(c) L^2 = \ell^2 + w^2 + h^2 \Rightarrow 2L \frac{dL}{dt} = 2\ell \frac{d\ell}{dt} + 2w \frac{dw}{dt} + 2h \frac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \Rightarrow dL/dt = 0 \text{ m/s}.$$

$$32. I = \frac{V}{R} \Rightarrow \frac{dI}{dt} = \frac{\partial I}{\partial V} \frac{dV}{dt} + \frac{\partial I}{\partial R} \frac{dR}{dt} = \frac{1}{R} \frac{dV}{dt} - \frac{V}{R^2} \frac{dR}{dt} = \frac{1}{R} \frac{dV}{dt} - \frac{I}{R} \frac{dR}{dt} \\ = \frac{1}{400} (-0.01) - \frac{0.03}{400} (0.03) = -0.000031 \text{ A/s}$$

$$33. \frac{dP}{dt} = 0.05, \frac{dT}{dt} = 0.15, V = 8.31 \frac{T}{P} \text{ and } \frac{dV}{dt} = \frac{8.31}{P} \frac{dT}{dt} - 8.31 \frac{T}{P^2} \frac{dP}{dt}. \text{ Thus when } P = 20 \text{ and } T = 320, \\ \frac{dV}{dt} = 8.31 \left[ \frac{0.15}{20} - \frac{(0.05)(320)}{400} \right] \approx -0.27 \text{ L/s}.$$

34. Let  $x$  and  $y$  be the respective distances of car  $A$  and car  $B$  from the intersection and let  $z$  be the distance between the two cars. Then  $dx/dt = -90$ ,  $dy/dt = -80$  and  $z^2 = x^2 + y^2$ . When  $x = 0.3$  and  $y = 0.4$ ,  $z = \sqrt{0.25} = 0.5$  and  $2z (dz/dt) = 2x (dx/dt) + 2y (dy/dt)$  or  $dz/dt = 0.6(-90) + 0.8(-80) = -118 \text{ km/h}$ .

35. (a) By the Chain Rule,  $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$ ,  $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} r \cos \theta$ .

$$(b) \left( \frac{\partial z}{\partial r} \right)^2 = \left( \frac{\partial z}{\partial x} \right)^2 \cos^2 \theta + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \cos \theta \sin \theta + \left( \frac{\partial z}{\partial y} \right)^2 \sin^2 \theta, \\ \left( \frac{\partial z}{\partial \theta} \right)^2 = \left( \frac{\partial z}{\partial x} \right)^2 r^2 \sin^2 \theta - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} r^2 \cos \theta \sin \theta + \left( \frac{\partial z}{\partial y} \right)^2 r^2 \cos^2 \theta. \text{ Thus} \\ \left( \frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial z}{\partial \theta} \right)^2 = \left[ \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \right] (\cos^2 \theta + \sin^2 \theta) = \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2.$$

36. By the Chain Rule,  $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} e^s \cos t + \frac{\partial u}{\partial y} e^s \sin t$ ,  $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} (-e^s \sin t) + \frac{\partial u}{\partial y} e^s \cos t$ . Then

$$\left( \frac{\partial u}{\partial s} \right)^2 = \left( \frac{\partial u}{\partial x} \right)^2 e^{2s} \cos^2 t + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} e^{2s} \cos t \sin t + \left( \frac{\partial u}{\partial y} \right)^2 e^{2s} \sin^2 t \text{ and} \\ \left( \frac{\partial u}{\partial t} \right)^2 = \left( \frac{\partial u}{\partial x} \right)^2 e^{2s} \sin^2 t - 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} e^{2s} \cos t \sin t + \left( \frac{\partial u}{\partial y} \right)^2 e^{2s} \cos^2 t. \text{ Thus} \\ \left[ \left( \frac{\partial u}{\partial s} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right] e^{-2s} = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2.$$



41.  $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} 2s + \frac{\partial z}{\partial y} 2r$ . Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r \partial s} &= \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} 2s \right) + \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} 2r \right) \\ &= \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial r} 2s + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \frac{\partial y}{\partial r} 2s + \frac{\partial z}{\partial x} \frac{\partial}{\partial r} 2s + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial r} 2r + \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \frac{\partial x}{\partial r} 2r + \frac{\partial z}{\partial y} 2 \\ &= 4rs \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y \partial x} 4s^2 + 0 + 4rs \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} 4r^2 + 2 \frac{\partial z}{\partial y}\end{aligned}$$

By the continuity of the partials,  $\frac{\partial^2 z}{\partial r \partial s} = 4rs \frac{\partial^2 z}{\partial x^2} + 4rs \frac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2) \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial z}{\partial y}$ .

42. By the Chain Rule,

(a)  $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$

(b)  $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta$

(c)  $\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial^2 z}{\partial \theta \partial r} = \frac{\partial}{\partial \theta} \left( \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta \right)$

$$\begin{aligned}&= -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \frac{\partial}{\partial \theta} \left( \frac{\partial z}{\partial x} \right) + \cos \theta \frac{\partial z}{\partial y} + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{\partial z}{\partial y} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial \theta} \right) + \cos \theta \frac{\partial z}{\partial y} + \sin \theta \left( \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial \theta} + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial \theta} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} + \cos \theta \left( -r \sin \theta \frac{\partial^2 z}{\partial x^2} + r \cos \theta \frac{\partial^2 z}{\partial y \partial x} \right) + \cos \theta \frac{\partial z}{\partial y} \\ &\quad + \sin \theta \left( r \cos \theta \frac{\partial^2 z}{\partial y^2} - r \sin \theta \frac{\partial^2 z}{\partial x \partial y} \right) \\ &= -\sin \theta \frac{\partial z}{\partial x} - r \cos \theta \sin \theta \frac{\partial^2 z}{\partial x^2} + r \cos^2 \theta \frac{\partial^2 z}{\partial y \partial x} + \cos \theta \frac{\partial z}{\partial y} \\ &\quad + r \cos \theta \sin \theta \frac{\partial^2 z}{\partial y^2} - r \sin^2 \theta \frac{\partial^2 z}{\partial y \partial x} \\ &= \cos \theta \frac{\partial z}{\partial y} - \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \sin \theta \left( \frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} \right) + r (\cos^2 \theta - \sin^2 \theta) \frac{\partial^2 z}{\partial y \partial x}\end{aligned}$$

43.  $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$  and  $\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta$ . Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos \theta \left( \frac{\partial^2 z}{\partial x^2} \cos \theta + \frac{\partial^2 z}{\partial y \partial x} \sin \theta \right) + \sin \theta \left( \frac{\partial^2 z}{\partial y^2} \sin \theta + \frac{\partial^2 z}{\partial x \partial y} \cos \theta \right) \\ &= \cos^2 \theta \frac{\partial^2 z}{\partial x^2} + 2 \cos \theta \sin \theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 z}{\partial \theta^2} &= -r \cos \theta \frac{\partial z}{\partial x} + (-r \sin \theta) \left( \frac{\partial^2 z}{\partial x^2} (-r \sin \theta) + \frac{\partial^2 z}{\partial y \partial x} r \cos \theta \right) \\ &\quad - r \sin \theta \frac{\partial z}{\partial y} + r \cos \theta \left( \frac{\partial^2 z}{\partial y^2} r \cos \theta + \frac{\partial^2 z}{\partial x \partial y} (-r \sin \theta) \right) \\ &= -r \cos \theta \frac{\partial z}{\partial x} - r \sin \theta \frac{\partial z}{\partial y} + r^2 \sin^2 \theta \frac{\partial^2 z}{\partial x^2} - 2r^2 \cos \theta \sin \theta \frac{\partial^2 z}{\partial x \partial y} + r^2 \cos^2 \theta \frac{\partial^2 z}{\partial y^2} \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r} &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2 z}{\partial x^2} + (\sin^2 \theta + \cos^2 \theta) \frac{\partial^2 z}{\partial y^2} - \frac{1}{r} \cos \theta \frac{\partial z}{\partial x} \\ &\quad - \frac{1}{r} \sin \theta \frac{\partial z}{\partial y} + \frac{1}{r} \left( \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right) \\ &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.} \end{aligned}$$

44. (a)
- $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$
- . Then

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \right) \\ &= \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2} \frac{\partial z}{\partial x} + \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial y} \right) \frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2} \frac{\partial z}{\partial y} \\ &= \frac{\partial^2 z}{\partial x^2} \left( \frac{\partial x}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2} \left( \frac{\partial y}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial y}{\partial t} \frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial t} \\ &= \frac{\partial^2 z}{\partial x^2} \left( \frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2} \left( \frac{\partial y}{\partial t} \right)^2 + \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial t} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{\partial^2 z}{\partial s \partial t} &= \frac{\partial}{\partial s} \left( \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \right) \\ &= \left( \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y \partial x} \frac{\partial y}{\partial s} \right) \frac{\partial x}{\partial t} + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial s \partial t} + \left( \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial s} \right) \frac{\partial y}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial s \partial t} \\ &= \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial s} \frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y} \left( \frac{\partial y}{\partial s} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} \frac{\partial x}{\partial s} \right) + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial s \partial t} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial s \partial t} + \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial s} \frac{\partial y}{\partial t} \end{aligned}$$

- 45.
- $F(x, y, z) = 0$
- is assumed to define
- $z$
- as a function of
- $x$
- and
- $y$
- , that is,
- $z = f(x, y)$
- . So by (7),
- $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$
- since
- $F_z \neq 0$
- . Similarly, it is assumed that
- $F(x, y, z) = 0$
- defines
- $x$
- as a function of
- $y$
- and
- $z$
- , that is
- $x = h(y, z)$
- . Then
- $F(h(y, z), y, z) = 0$
- and by the Chain Rule,
- $F_x \frac{\partial x}{\partial y} + F_y \frac{\partial y}{\partial y} + F_z \frac{\partial z}{\partial y} = 0$
- . But
- $\frac{\partial z}{\partial y} = 0$
- and
- $\frac{\partial y}{\partial y} = 1$
- , so

$$F_x \frac{\partial x}{\partial y} + F_y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{F_y}{F_x}. \text{ A similar calculation shows that } \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}. \text{ Thus}$$

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = \left( -\frac{F_x}{F_z} \right) \left( -\frac{F_y}{F_x} \right) \left( -\frac{F_z}{F_y} \right) = -1.$$