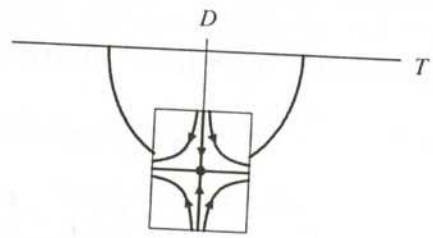
we have a sink with repeated eigenvalues. If $-4 < a < 0$, we have complex eigenvalues with negative real parts. Therefore, the phase portraits are spiral sinks. If $a = 0$, we have a degenerate case with an entire line of equilibrium points. Finally, if $a > 0$, the corresponding portion of the line is below the T-axis, and the phase portraits are saddles.

(c) Bifurcations occur at $a = -4$, where we have a sink with repeated eigenvalues, and at $a = 0$, where we have zero as a repeated eigenvalue. For $a = 0$, the y-axis is entirely composed of equilibrium points.

5. **(a)**

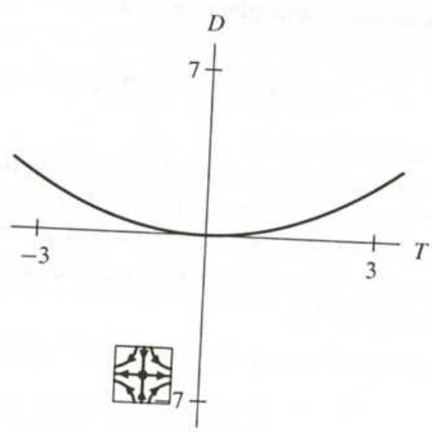


(b) The curve in the trace-determinant plane is the portion of the unit circle centered at 0 that lies in the half-plane $y \leq 0$.

A glance at the trace-determinant plane shows that for $-1 < a < 1$, we have a saddle. If $a = 1$, the eigenvalues are 0 and 1. If $a = -1$, the eigenvalues are 0 and -1.

(c) Bifurcations occur only at $a = \pm 1$. For these two special values of a, we have a line of equilibrium points. The nonzero equilibrium points disappear if $-1 < a < 1$.

6. **(a)**



(b) The curve in the trace-determinant plane is not a curve at all. For all values of a, $T = -1$ and $D = -6$. So the curve is simply a point in the trace-determinant plane. For all a, we have a saddle.

(c) There are no bifurcations, since the origin is always a saddle. (There is nothing special about $a = 0$, by the way.)

If $(a-1)^2+4b<0$, we have complex eigenvalues with real parts $(a+1)/2$. So if $a<-1$, we have a spiral sink; if $a>-1$, we have a spiral source; and if $a=-1$, we have a center.

The systems with zero determinant for this family satisfies $a=b$ since $D=a-b$. If $a>b$, $D>0$, and if $a<b$, we have $D<0$. So in the case of real eigenvalues ($(a-1)^2+4b>0$), we have a saddle if $a<b$ because $D<0$. If we graph the line $b=a$ together with the parabola $(a-1)^2+4b=0$, we see that they are tangent at the point $(-1,-1)$. The regions between the line $a=b$ and the parabola $(a-1)^2+4b=0$ give the places where we have sinks or sources with real eigenvalues. If $a>-1$ in this region, then both eigenvalues are positive, so we have a source. If $a<-1$ in this region, then both eigenvalues are negative (a sink). If $(a,b)=(-1,-1)$, we have repeated zero eigenvalues. Whew! That was a toughie! It is worthwhile to draw a picture of the ab-plane.

9. The eigenvalues are roots of the equation $\lambda^2-2a\lambda+a^2-b^2=0$. These roots are

$$a\pm\sqrt{b^2}=a\pm|b|.$$

So we have a repeated zero eigenvalue if $a=b=0$.

If $a=\pm b$, then one of the eigenvalues is 0, and as long as $a\neq 0$ (so $b\neq 0$), the other eigenvalue is nonzero.

The eigenvalues are repeated (both equal to a) if $b=0$. The eigenvalues are never complex since $\sqrt{b^2}\geq 0$.

If $a>|b|$, then $a\pm|b|>0$, so we have a source with real eigenvalues. If $a<0$ and $-a>|b|$, then $a\pm|b|<0$, so we have a sink with real eigenvalues. In all other cases we have a saddle.

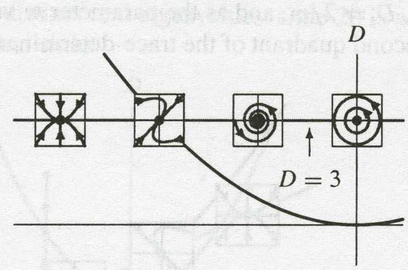
10. The eigenvalues are roots of the equation $\lambda^2-2a\lambda+a^2+b^2=0$. They are $a\pm ib$. Hence we have complex roots if $b\neq 0$. If $b\neq 0$ and $a<0$, the phase portrait is a spiral sink; if $a=0$, the phase portrait is a center; and if $a>0$, the phase portrait is a spiral source. If $b=0$, a is a repeated eigenvalue (repeated 0 eigenvalue if $a=0$, source if $a>0$, and sink if $a<0$).

11. **(a)** This second-order equation is equivalent to the system

$$\frac{dy}{dt}=v$$
$$\frac{dv}{dt}=-3y-bv.$$

Therefore, $T=-b$ and $D=3$. So the corresponding curve in the trace-determinant plane is $D=3$.

(b)



(c) The line $D=3$ in the trace-determinant plane crosses the repeated-eigenvalue parabola $D=T^2/4$ if $b^2=12$, which implies that $b=2\sqrt{3}$ since b is a nonnegative parameter. If

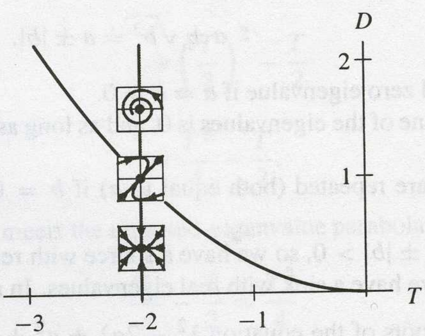
$b = 0$, we have pure imaginary eigenvalues—the undamped case. If $0 < b < 2\sqrt{3}$, the eigenvalues are complex with a negative real part—the underdamped case. If $b = 2\sqrt{3}$, the eigenvalues are repeated and negative—the critically damped case. Finally, if $b > 2\sqrt{3}$, the eigenvalues are real and negative—the overdamped case.

12. **(a)** This second-order equation is equivalent to the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -ky - 2v.$$

Therefore, $T = -2$ and $D = k$. So the curve in the trace-determinant plane is the vertical line $T = -2$.

(b)



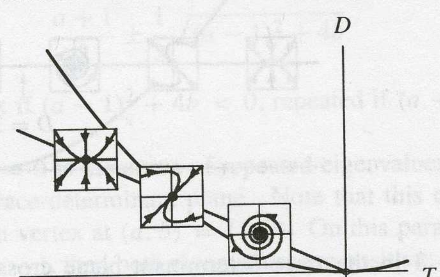
(c) The line $T = -2$ meets the parabola $D = T^2/4$ at $(T, D) = (-2, 1)$, which corresponds to $k = 1$. From the trace-determinant plane, we see that we have a sink with real eigenvalues if $0 < k \leq 1$, repeated eigenvalues if $k = 1$, and complex eigenvalues if $k > 1$. Therefore, the oscillator is overdamped if $0 < k \leq 1$, critically damped if $k = 1$, and underdamped if $k > 1$.

13. **(a)** The second-order equation reduces to the first-order system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -\frac{2}{m}y + \frac{1}{m}v.$$

Hence $T = -1/m$, $D = 2/m$, and as the parameter m varies, the systems move along the line $D = -2T$ in the second quadrant of the trace-determinant plane.

(b)



If $0 < b < 2\sqrt{3}$, the
case. If $b = 2\sqrt{3}$, the
nally, if $b > 2\sqrt{3}$, the

plane is the vertical line

, which corresponds to
with real eigenvalues if
$k > 1$. Therefore, the
nderdamped if $k > 1$.

ms move along the line

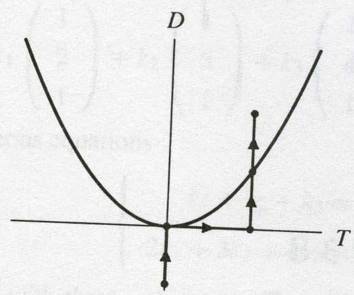
(c) The line $D = -2T$ intersects the repeated-eigenvalue parabola $D = T^2/4$ at the point (T, D) that satisfies $-2T = T^2/4$. We have

$$\frac{T^2}{4} + 2T = T\left(\frac{T}{4} + 2\right) = 0,$$

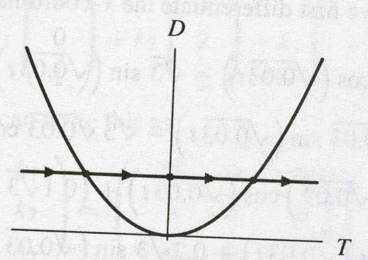
which yields $T = 0$ or $T = -8$.

For $-8 < T < 0$, the system is underdamped; for $T = -8$, the system is critically damped; and for $T < -8$, the system is overdamped. Since $T = -1/m$, the system is overdamped if $0 < m < 1/8$; it is critically damped if $m = 1/8$; and it is underdamped if $m > 1/8$.

14. (a) In Animation A, slides 0–11 are saddles, and slides 12–20 include a line of equilibrium points. Slides 21–23 are sources with distinct real eigenvalues, slide 24 is a source with repeated eigenvalues, and slides 25–32 are spiral sources.



(b) In Animation B, slides 0–7 are sinks with distinct real eigenvalues, slide 8 is a sink with repeated eigenvalues, and slides 9–15 are spiral sinks. Slide 16 is a center. Then slides 17–23 are spiral sources, slide 24 is a source with repeated eigenvalues, and slides 25–32 are sources with distinct real eigenvalues.



(c) In Animation C, slides 0–15 are saddles, and slide 16 includes a line of equilibrium points. Slides 17–19 are sinks with distinct real eigenvalues, slide 20 is a sink with repeated eigenvalues, and slides 21–32 are spiral sinks.