In matrix notation, this system is

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{Y} = \begin{pmatrix} y \\ v \\ w \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r & -p & -q \end{pmatrix}.$$

20. If there are more than the usual number of buyers, then $b > 0$. If this level of buying means that prices will increase and that fewer buyers will enter the market, then the effect on db/dt should be negative. Since $db/dt = \alpha b + \beta s$, we expect that the αb-term will be negative if $b > 0$. Consequently, α should be negative.

21. If there are fewer than the usual number of buyers, then $b < 0$. If this level of b has a negative effect on the number of sellers, we expect the γb-term in ds/dt to be negative. If $\gamma b < 0$ and $b < 0$, then we must have $\gamma > 0$.

22. If $s > 0$, there are more than the usual number of houses for sale and house prices should decline. Declining prices should have a positive effect on the number of buyers and a negative effect on the number of sellers. Since $db/dt = \alpha b + \beta s$, we expect the βs-term to be positive. Since $\beta s > 0$ if $s > 0$, the parameter β should be positive.

23. In the model, $ds/dt = \gamma b + \delta s$. If $s > 0$, then the number of sellers is greater than usual and house prices should decline. Since declining prices should have a negative effect on the number of sellers, we expect the δs-term to be negative. If $\delta s < 0$ when $s > 0$, we should have $\delta < 0$.

24. **(a)** Substituting $\mathbf{Y}_1(t)$ in the left-hand side of the differential equation yields

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Moreover, the right-hand side becomes

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}_1(t) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ e^t \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Since the two sides of the differential equation agree, $\mathbf{Y}_1(t)$ is a solution.

Similarly, if we substitute $\mathbf{Y}_2(t)$ in the left-hand side of the differential equation, we get

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

Moreover, the right-hand side is

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{Y}_2(t) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

34. **(a)** If $\mathbf{Y}(t) = (t, t^2/2)$, then $x(t) = t$ and $y(t) = t^2/2$. Then $dx/dt = 1$, and $dy/dt = t = x$. So $\mathbf{Y}(t)$ satisfies the differential equation.

(b) For $2\mathbf{Y}(t)$, we have $x(t) = 2t$, and $y(t) = t^2$. In this case, we need only consider $dx/dt = 2$ to see that the function is not a solution to the system.

35. **(a)** Using the Product Rule we compute

$$\frac{dW}{dt} = \frac{dx_1}{dt}y_2 + x_1\frac{dy_2}{dt} - \frac{dx_2}{dt}y_1 - x_2\frac{dy_1}{dt}.$$

(b) Since $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions, we know that

$$\frac{dx_1}{dt} = ax_1 + by_1$$
$$\frac{dy_1}{dt} = cx_1 + dy_1$$

and that

$$\frac{dx_2}{dt} = ax_2 + by_2$$
$$\frac{dy_2}{dt} = cx_2 + dy_2.$$

Substituting these equations into the expression for dW/dt, we obtain

$$\frac{dW}{dt} = (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1).$$

After we collect terms, we have

$$\frac{dW}{dt} = (a + d)W.$$

(c) This equation is a homogeneous, linear, first-order equation (as such it is also separable—see Sections 1.1, 1.2, and 1.8). Therefore, we know that the general solution is

$$W(t) = Ce^{(a+d)t}$$

where C is any constant (but note that $C = W(0)$).

(d) From Exercises 31 and 32, we know that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent if and only if $W(t) \neq 0$. But, $W(t) = Ce^{(a+d)t}$, so $W(t) = 0$ if and only if $C = W(0) = 0$. Hence, $W(t) = 0$ is zero for some t if and only if $C = W(0) = 0$.

EXERCISES FOR SECTION 3.2

1. **(a)** The characteristic polynomial is

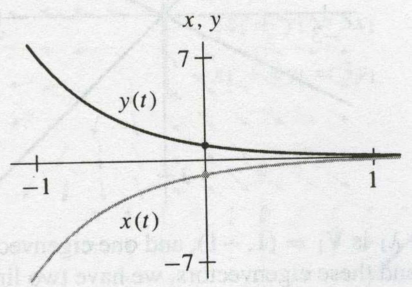
$$(3 - \lambda)(-2 - \lambda) = 0,$$

and therefore the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$.

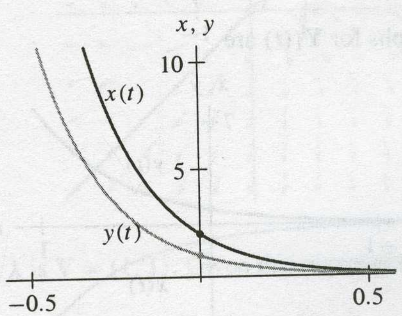
Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$ are



and the $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$ are



(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

3. **(a)** The characteristic polynomial is

$$(4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0,$$

and therefore the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$.

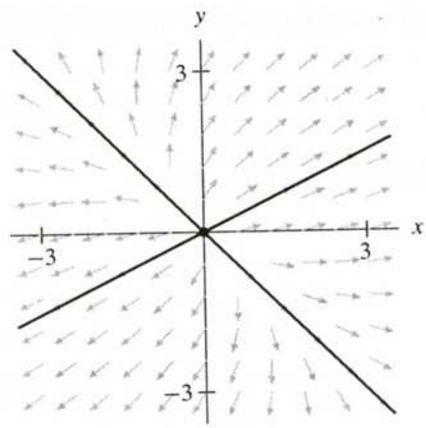
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 2$, we solve the system of equations

$$\begin{cases} 4x_1 + 2y_1 = 2x_1 \\ x_1 + 3y_1 = 2y_1 \end{cases}$$

and obtain $y_1 = -x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2 = 2y_2$ for $\lambda_2 = 5$.

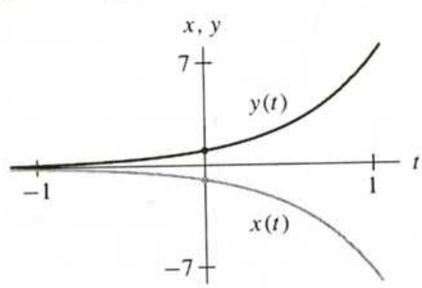
(c)



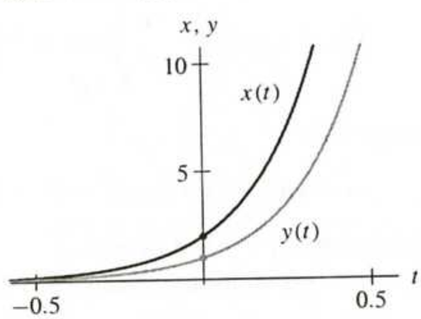
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (1, -1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (2, 1)$. Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$ are



and the $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$ are



(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (-2, 1)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 5k_1 = -2 \\ k_1 + k_2 = 1. \end{cases}$$

Solving these equations, we obtain $k_1 = -2/5$ and $k_2 = 7/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = -\tfrac{2}{5}e^{-3t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \tfrac{7}{5}e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, $x(t) = -2e^{-3t}$ and $y(t) = (-2e^{-3t} + 7e^{2t})/5$.

12. Note that this system is just the negative of the one in the Exercise 11. Hence, to solve the first two initial-value problems, we could just take the corresponding solutions from that problem and run them backwards. That is, replace t by $-t$. However, just for kicks, we will solve this exercise without using the result of Exercise 11.

The characteristic polynomial is

$$(3 - \lambda)(-2 - \lambda) = 0,$$

and therefore the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$.

To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 3$, we solve the system of equations

$$\begin{cases} 3x_1 = 3x_1 \\ x_1 - 2y_1 = 3y_1 \end{cases}$$

and obtain

$$5y_1 = x_1.$$

Therefore, an eigenvector for the eigenvalue $\lambda_1 = 3$ is $\mathbf{V}_1 = (5, 1)$.

Using the same procedure, we obtain the eigenvector $\mathbf{V}_2 = (0, 1)$ for $\lambda_2 = -2$.

The general solution to this linear system is therefore

$$\mathbf{Y}(t) = k_1 e^{3t} \begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) We have $\mathbf{Y}(0) = (1, 0)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 5k_1 = 1 \\ k_1 + k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 1/5$ and $k_2 = -1/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{1}{5}e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} - \tfrac{1}{5}e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) We have $\mathbf{Y}(0) = (0, 1)$. Since this initial condition is an eigenvector associated to the $\lambda = -2$ eigenvalue, we do not need to do any additional calculation. The desired solution to the initial-value problem is

$$\mathbf{Y}(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (2, 2)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 5k_1 = 2 \\ k_1 + k_2 = 2. \end{cases}$$

Solving these equations, we obtain $k_1 = 2/5$ and $k_2 = 8/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{2}{5}e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \tfrac{8}{5}e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

13. The characteristic polynomial is

$$(-4 - \lambda)(-3 - \lambda) - 2 = \lambda^2 + 7\lambda + 10 = 0,$$

and therefore the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = -2$.

To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -5$, we solve the system of equations

$$\begin{cases} -4x_1 + y_1 = -5x_1 \\ 2x_1 - 3y_1 = -5y_1 \end{cases}$$

and obtain

$$y_1 = -x_1.$$

Therefore, an eigenvector for the eigenvalue $\lambda_1 = -5$ is $\mathbf{V}_1 = (1, -1)$.

Using the same procedure, we obtain the eigenvector $\mathbf{V}_2 = (1, 2)$ for $\lambda_2 = -2$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) We have $\mathbf{Y}(0) = (1, 0)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} k_1 + k_2 = 1 \\ -k_1 + 2k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 2/3$ and $k_2 = 1/3$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{2}{3}e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \tfrac{1}{3}e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(b) We have $\mathbf{Y}(0) = (2, 1)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} k_1 + k_2 = 2 \\ -k_1 + 2k_2 = 1. \end{cases}$$

Solving these equations, we obtain $k_1 = 1$ and $k_2 = 1$. Thus, the particular solution is

$$\mathbf{Y}(t) = e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (-1, -2)$. Since this initial condition is an eigenvector associated to the $\lambda = -2$ eigenvalue, we do not need to do any additional calculation. The desired solution to the initial-value problem is

$$\mathbf{Y}(t) = -e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (-1, -2)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 2k_1 + k_2 = -1 \\ k_1 + k_2 = -2. \end{cases}$$

Solving these equations, we obtain $k_1 = 1$ and $k_2 = -3$. Thus, the particular solution is

$$\mathbf{Y}(t) = e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

15. Given any vector $\mathbf{Y}_0 = (x_0, y_0)$, we have

$$\mathbf{A}\mathbf{Y}_0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} ax_0 \\ ay_0 \end{pmatrix} = a \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = a\mathbf{Y}_0.$$

Therefore, every nonzero vector is an eigenvector associated to the eigenvalue a.

16. The characteristic polynomial of $\mathbf{A}$ is

$$(a - \lambda)(d - \lambda) = 0,$$

and thus the eigenvalues of $\mathbf{A}$ are $\lambda_1 = a$ and $\lambda_2 = d$.

To find the eigenvectors $\mathbf{V}_1 = (x_1, y_1)$ associated to $\lambda_1 = a$, we need to solve the equation

$$\mathbf{A}\mathbf{V}_1 = a\mathbf{V}_1$$

for all possible vectors $\mathbf{V}_1$. Rewritten in terms of components, this equation is equivalent to

$$\begin{cases} ax_1 + by_1 = ax_1 \\ dy_1 = ay_1. \end{cases}$$

Since $a \neq d$, the second equation implies that $y_1 = 0$. If so, then the first equation is satisfied for all x_1. In other words, the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue a are the vectors of the form $(x_1, 0)$.

To find the eigenvectors $\mathbf{V}_2 = (x_2, y_2)$ associated to $\lambda_2 = d$, we need to solve the equation

$$\mathbf{A}\mathbf{V}_2 = d\mathbf{V}_2$$

for all possible vectors $\mathbf{V}_2$. Rewritten in terms of components, this equation is equivalent to

$$\begin{cases} ax_2 + by_2 = dx_2 \\ dy_2 = dy_2. \end{cases}$$

The second equation always holds, so the eigenvectors $\mathbf{V}_2$ are those vectors that satisfy the equation $ax_2 + by_2 = dx_2$, which can be rewritten as

$$by_2 = (d - a)x_2.$$

These vectors form a line through the origin of slope $(d - a)/b$.

17. The characteristic polynomial of $\mathbf{B}$ is

$$\lambda^2 - (a + d)\lambda + (a \cdot d - b \cdot b)$$

or

$$\lambda^2 - (a + d)\lambda + ad - b^2.$$

The roots of this polynomial are

$$\begin{aligned} \lambda &= \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - b^2)}}{2} \\ &= \frac{a + d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2} \\ &= \frac{a + d \pm \sqrt{(a-d)^2 + 4b^2}}{2} \end{aligned}$$

Since the discriminant $D = (a-d)^2 + 4b^2$ is always nonnegative, the roots λ are real. Therefore, the matrix $\mathbf{B}$ has real eigenvalues. If $b \neq 0$, then D is positive and hence $\mathbf{B}$ has two distinct eigenvalues. (The only way to have only one eigenvalue is for $D = 0$).

18. The characteristic equation is

$$(a - \lambda)(-\lambda) - bc = \lambda^2 - a\lambda - bc = 0.$$

Finding the roots via the quadratic formula, we obtain the eigenvalues

$$\frac{a \pm \sqrt{a^2 + 4bc}}{2}.$$

Note that these eigenvalues are very different from the case where the matrix is upper triangular (see Exercise 16). For example, they are not necessarily real numbers because $a^2 + 4bc$ can be negative.

19. **(a)** To form the system, we introduce the new dependent variable $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -p\frac{dy}{dt} - qy = -pv - qy.$$

Written in matrix form this system where $\mathbf{Y} = (y, v)$, we have

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{Y}.$$

(b) The characteristic polynomial is

$$(0 - \lambda)(-p - \lambda) + q = \lambda^2 + p\lambda + q.$$

(c) The roots of this polynomial (the eigenvalues) are

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

(d) The roots are distinct real numbers if the discriminant $D = p^2 - 4q$ is positive. In other words, the roots are distinct real numbers if $p^2 > 4q$.

(e) Since q is positive, $p^2 - 4q < p^2$, so we know that $\sqrt{p^2 - 4q} < \sqrt{p^2} = p$. Since the numerator in the expression for the eigenvalues is $-p \pm \sqrt{p^2 - 4q}$, we see that it must be negative. Since the denominator is positive, the eigenvalues must be negative.

20. **(a)** The parameters $m = 1$, $k = 4$, and $b = 5$ yield the second-order equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0.$$

Given $v = dy/dt$, the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -4y - 5v.$$

The characteristic polynomial is $\lambda^2 + 5\lambda + 4$, and the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -1$. To find the eigenvectors $\mathbf{V}_1 = (y_1, v_1)$ associated to the eigenvalue $\lambda_1 = -4$, we solve the system of equations.

$$\begin{cases} v_1 = -4y_1 \\ -4y_1 - 5v_1 = -4v_1 \end{cases}$$

and obtain $v_1 = -4y_1$. Thus, one eigenvector for $\lambda_1 = -4$ is $\mathbf{V}_1 = (1, -4)$. By the same procedure, we can find the eigenvector $\mathbf{V}_2 = (1, -1)$ for the eigenvalue $\lambda_2 = -1$.

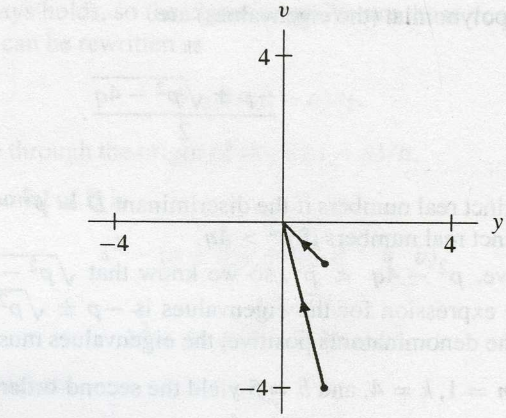
(b) Therefore the solution $\mathbf{Y}_1(t)$ that satisfies $\mathbf{Y}_1(0) = \mathbf{V}_1$ is

$$\mathbf{Y}_1(t) = e^{-4t}\begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

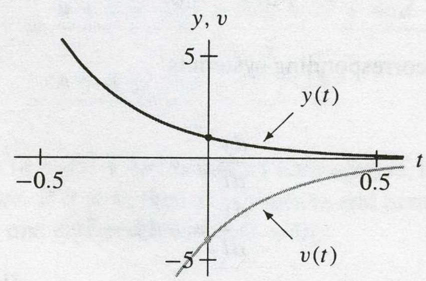
The solution $\mathbf{Y}_2(t)$ that satisfies $\mathbf{Y}_2(0) = \mathbf{V}_2$ is

$$\mathbf{Y}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

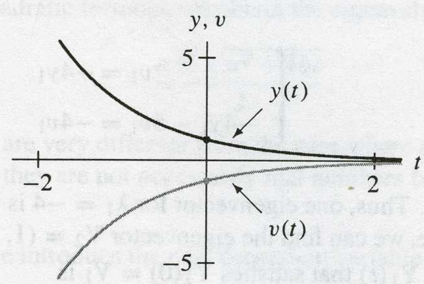
(c)



(d) The $y(t)$- and $v(t)$-graphs for $\mathbf{Y}_1(t)$ are



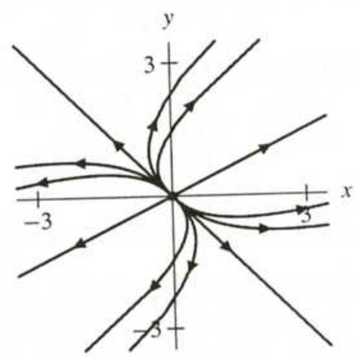
and the $y(t)$- and $v(t)$-graphs for $\mathbf{Y}_2(t)$ are



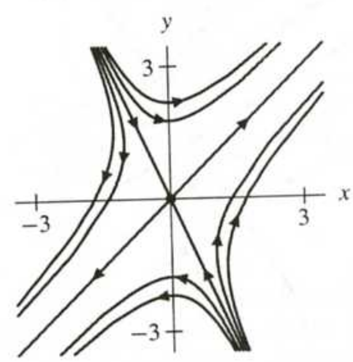
(e) The first initial condition $(y_0, v_0) = (1, -4)$ represents a solution whose initial position is 1 unit away from the equilibrium position and whose initial velocity is -4. Note that the solution tends toward the equilibrium point at the origin. Moreover, $y(t)$ is decreasing toward 0, and $v(t)$ is increasing toward 0. Therefore, the mass moves toward the equilibrium position monotonically, and its speed decreases as it approaches the equilibrium position. The mass does not oscillate about the equilibrium position.

The second initial condition $(y_0, v_0) = (1, -1)$ represents a solution whose initial position is 1 unit away from the equilibrium position and whose initial velocity is -1. The behavior of this solution is similar to the first solution.

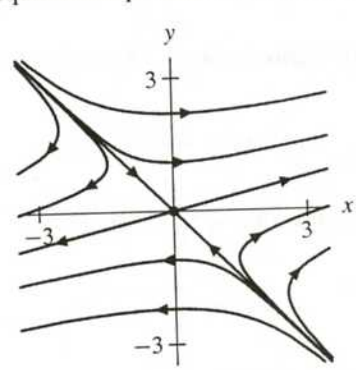
3. As we computed in Exercise 3 of Section 3.2, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$. The eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 2$ satisfy $y_1 = -x_1$, and the eigenvectors (x_2, y_2) for $\lambda_2 = 5$ satisfy $x_2 = 2y_2$. The equilibrium point at the origin is a source.



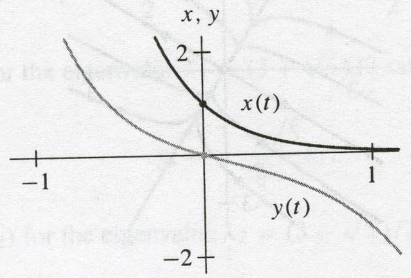
4. As we computed in Exercise 6 of Section 3.2, the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 9$. The eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -4$ satisfy $9x_1 = -4y_1$, and the eigenvectors (x_2, y_2) for $\lambda_2 = 9$ satisfy the equation $y_2 = x_2$. The equilibrium point at the origin is a saddle.



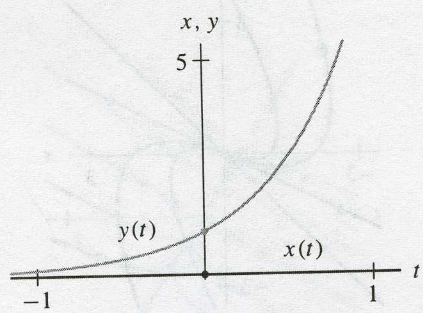
5. As we computed in Exercise 7 of Section 3.2, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$. The eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -1$ satisfy $y_1 = -x_1$, and the eigenvectors (x_2, y_2) for $\lambda_2 = 4$ satisfy $x_2 = 4y_2$. The equilibrium point at the origin is a saddle.



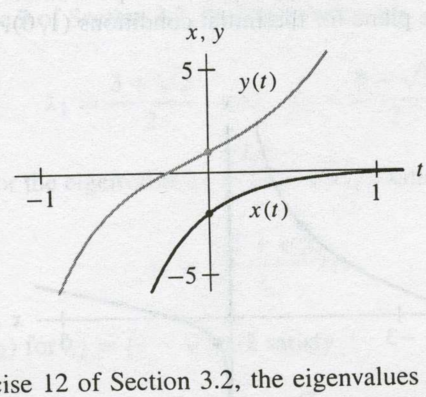
(a) The solution curve with initial condition $(1, 0)$ is asymptotic to the negative y-axis as $t \to \infty$ and is asymptotic to the line $y = x/5$ in the first quadrant as $t \to -\infty$.



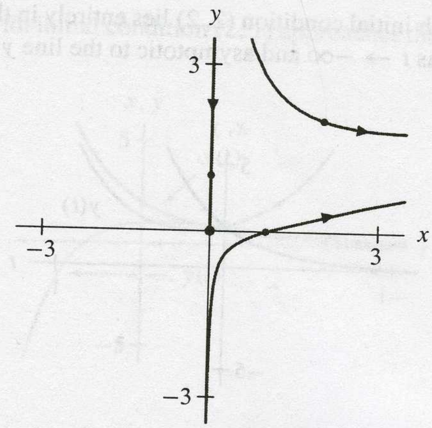
(b) The solution curve with initial condition $(0, 1)$ lies entirely on the positive y-axis, and $y(t) \to \infty$ in an exponential fashion as $t \to \infty$.



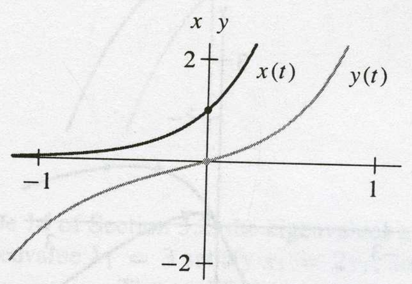
(c) The solution curve with initial condition $(-2, 1)$ is asymptotic to the positive y-axis as $t \to \infty$ and is asymptotic to the line $y = x/5$ in the third quadrant as $t \to -\infty$.



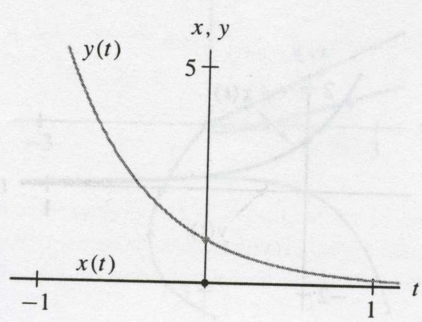
10. As we computed in Exercise 12 of Section 3.2, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$. The eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 3$ satisfy $y_1 = x_1/5$, and the eigenvectors (x_2, y_2) for the eigenvalue $\lambda_2 = -2$ satisfy $x_2 = 0$. The equilibrium point at the origin is a saddle. Therefore, the solution curves in the phase plane for the initial conditions $(1, 0)$, $(0, 1)$, and $(2, 2)$ are shown in the following figure.



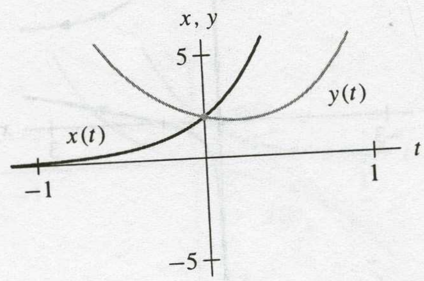
(a) The solution curve with initial condition (1, 0) is asymptotic to the negative y-axis as $t \to -\infty$ and is asymptotic to the line $y = x/5$ in the first quadrant as $t \to \infty$.



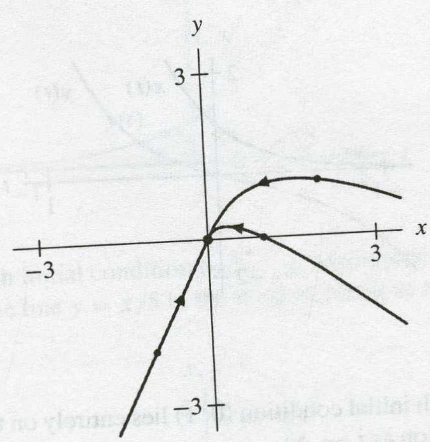
(b) The solution curve with initial condition (0, 1) lies entirely on the positive y-axis, and $y(t) \to 0$ in an exponential fashion as $t \to \infty$.



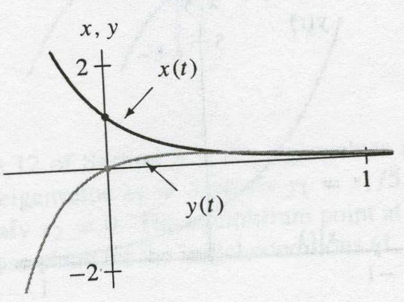
(c) The solution curve with initial condition $(2, 2)$ lies entirely in the first quadrant. It is asymptotic to the positive y-axis as $t \to -\infty$ and asymptotic to the line $y = x/5$ as $t \to \infty$.



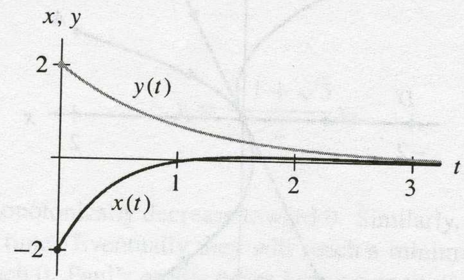
11. As we computed in Exercise 13 of Section 3.2, the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = -2$. The eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -5$ satisfy $y_1 = -x_1$, and the eigenvectors (x_2, y_2) for the eigenvalue $\lambda_2 = -2$ satisfy $y_2 = 2x_2$. The equilibrium point at the origin is a sink. The solution curves in the phase plane for the initial conditions $(1, 0)$, $(2, 1)$, and $(-1, -2)$ are shown in the following figure.



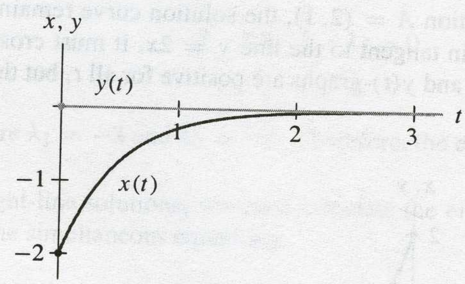
(a) The solution curve with initial condition $(1, 0)$ approaches the origin tangent to the line $y = 2x$.



For the initial condition $C = (-2, 2)$, we see that $y(t)$ is decreasing but positive for all also see that $x(t)$ is increasing initially. It becomes positive, reaches a maximum, an decreases as it approaches 0. Again, these two graphs do not cross at any time.



The initial condition $D = (-2, 0)$ lies on the line of eigenvectors associated to the eigen $\lambda_1 = -1$. Therefore, the solution curve remains on the x-axis for all t. Hence, $y(t) = 0$ fo t, and $x(t)$ is the exponential function $-2e^{-2t}$.



20. **(a)** The characteristic equation is

$$(2 - \lambda)(-2 - \lambda) - 12 = \lambda^2 - 16 = 0,$$

so the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 4$. Therefore, the equilibrium point at the origin is saddle.

(b) To find all the straight-line solutions, we must calculate the eigenvectors. For the eigenvalu $\lambda_1 = -4$, we have the simultaneous equations

$$\begin{cases} 2x_1 + 6y_1 = -4x_1 \\ 2x_1 - 2y_1 = -4y_1, \end{cases}$$

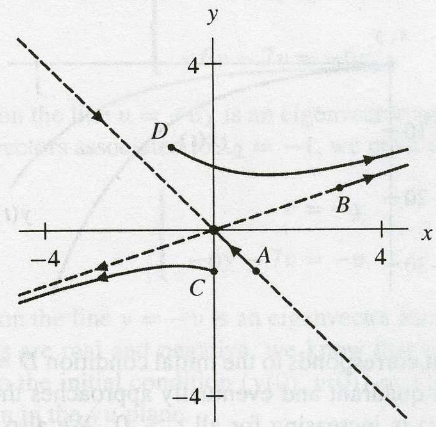
and we obtain $y_1 = -x_1$. In other words, all vectors on the line $y_1 = -x_1$ are eigenvectors for λ_1. Therefore, any solution of the form $e^{-4t}(x_1, -x_1)$ for any x_1 is a straight-line solution corresponding to the eigenvalue $\lambda_1 = -4$.

To calculate the eigenvectors associated to the eigenvalue $\lambda_2 = 4$, we must solve the equations

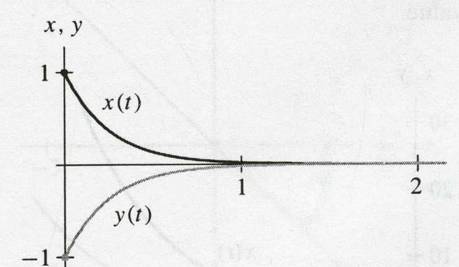
$$\begin{cases} 2x_2 + 6y_2 = 4x_2 \\ 2x_2 - 2y_2 = 4y_2, \end{cases}$$

and we obtain $x_2 = 3y_2$. Therefore, any solution of the form $e^{4t}(3y_2, y_2)$ for any y_2 is a straight-line solution corresponding to the eigenvalue $\lambda_2 = 4$.

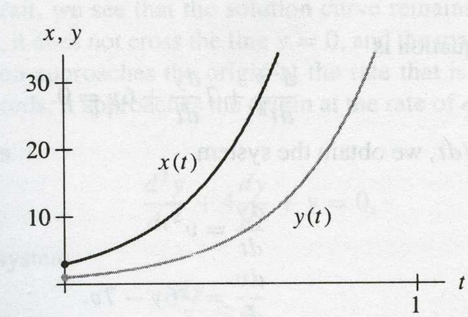
(c) In the phase plane, the only solution curves that approach the origin are those whose initial conditions lie on the line $y = -x$. All other solution curves eventually approach those that correspond to the line $x = 3y$.



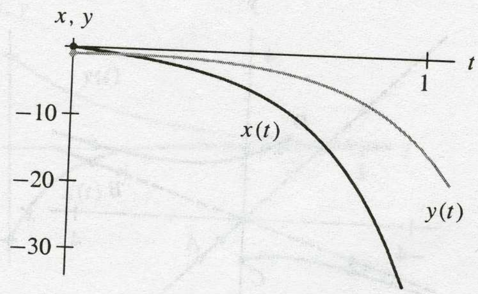
The initial condition $A = (1, -1)$ lies on the line $y = -x$. Therefore, it corresponds to a straight-line solution. In fact, the formula for its solution is $e^{-4t}(1, -1)$.



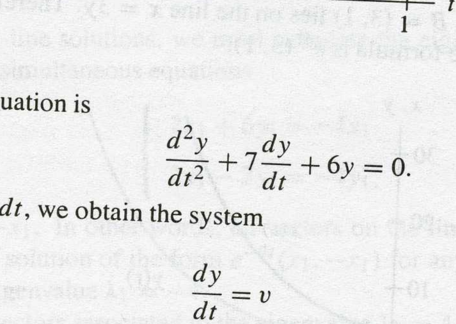
The initial condition $B = (3, 1)$ lies on the line $x = 3y$. Therefore, it corresponds to a straight-line solution, and the formula is $e^{4t}(3, 1)$.



The solution curve that corresponds to the initial condition $C = (0, -1)$ enters the third quadrant and eventually approaches line $x = 3y$. From the phase plane, we see that $x(t)$ is decreasing for all $t > 0$. We also see that $y(t)$ increases initially, reaches a negative maximum value, and then decreases in an exponential fashion. Since the solution curve crosses the line $y = x$, we know that these two graphs cross. By examining the line where $dy/dt = 0$, we see that these two graphs cross at precisely the same time as $y(t)$ attains its maximum value.



The solution curve that corresponds to the initial condition $D = (-1, 2)$ moves from the second quadrant into the first quadrant and eventually approaches the line $x = 3y$. From the phase plane, we see that $x(t)$ is increasing for all $t > 0$. We also see that $y(t)$ decreases initially, reaches a positive minimum value, and then increases in an exponential fashion. Since this solution curve crosses the line $y = x$, we know that these two graphs cross. By examining the line for which $dy/dt = 0$, we see that these two graphs cross at precisely the same time as $y(t)$ attains its minimum value.



21. **(a)** The second-order equation is

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 6y = 0.$$

Introducing $v = dy/dt$, we obtain the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -6y - 7v.$$

(b) The characteristic polynomial is

$$\lambda^2 + 7\lambda + 6,$$

which factors into $(\lambda + 6)(\lambda + 1)$.

(c) From the characteristic polynomial, we obtain the eigenvalues $\lambda_1 = -6$ and $\lambda_2 = -1$.

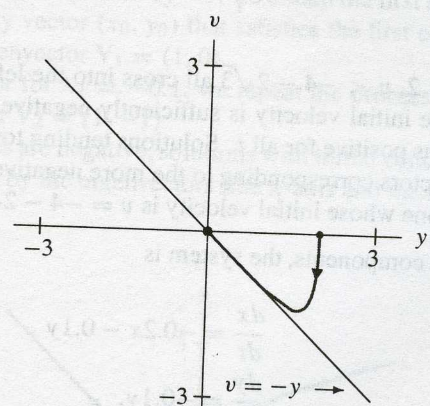
(d) To compute the eigenvectors associated to $\lambda_1 = -6$, we solve the simultaneous equations

$$\begin{cases} v = -6y \\ -6y - 7v = -6v. \end{cases}$$

Therefore, any vector on the line $v = -6y$ is an eigenvector associated to the eigenvalue λ_1. To compute the eigenvectors associated to $\lambda_2 = -1$, we must solve the simultaneous equations

$$\begin{cases} v = -y \\ -6y - 7v = -v. \end{cases}$$

Therefore, any vector on the line $y = -v$ is an eigenvector associated to the eigenvalue λ_2. Since both eigenvalues are real and negative, we know that origin is a sink, and the solution curve corresponding to the initial condition $(y(0), v(0)) = (2, 0)$ tends toward the origin tangent to the line $y = -v$ in the yv-plane.



From the phase portrait, we see that the solution curve remains in the fourth quadrant for all $t > 0$. Consequently, it does not cross the line $y = 0$, and the mass cannot cross the equilibrium position. The solution approaches the origin at the rate that is determined by the eigenvalue $\lambda_2 = -1$. In other words, it approaches the origin at the rate of e^{-t}.

22. The differential equation is

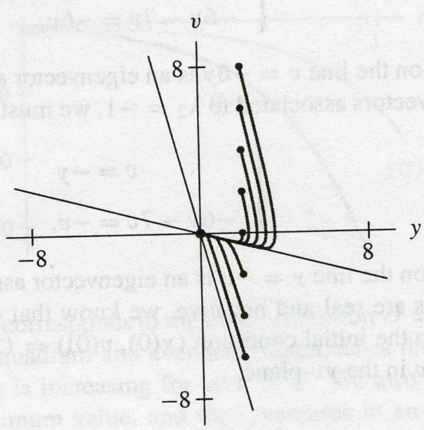
$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0,$$

which corresponds to the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y - 4v.$$

The characteristic polynomial is $\lambda^2 + 4\lambda + 1$, and consequently the eigenvalues are $\lambda = -2 \pm \sqrt{3}$.

The eigenvectors for $\lambda = -2 + \sqrt{3}$ satisfy $v = (-2 + \sqrt{3})y$, and the eigenvectors for $\lambda = -2 - \sqrt{3}$ satisfy $v = (-2 - \sqrt{3})y$. Looking at the phase plane, the line $y = 2$ crosses each line of eigenvectors once. The line of eigenvectors corresponding to $\lambda = -2 - \sqrt{3}$ is crossed at $v = -4 - 2\sqrt{3}$ while the line of eigenvectors corresponding to $\lambda = -2 + \sqrt{3}$ is crossed at $v = -4 + 2\sqrt{3}$.



The solutions with $y = 2$, $v < -4 - 2\sqrt{3}$ all cross into the left-half ($y < 0$ half) of the phase plane. In other words, if the initial velocity is sufficiently negative, then y overshoots $y = 0$. For $v \geq -4 - 2\sqrt{3}$, $y(t)$ remains positive for all t. Solutions tending toward the origin most quickly are those on the line of eigenvectors corresponding to the more negative eigenvalue, so the solution that reaches 0.1 quickest is the one whose initial velocity is $v = -4 - 2\sqrt{3}$.

23. **(a)** Written in terms of its components, the system is

$$\frac{dx}{dt} = -0.2x - 0.1y$$
$$\frac{dy}{dt} = -0.1y.$$

Since the coefficient of y in dx/dt is negative, the introduction of new fish ($y > 0$) contributes negatively to dx/dt. Hence, the new fish have a negative affect on the native fish population. Since the equation for dy/dt involves only y, the native fish have no affect on the population of the new fish.

(b) If the new species of fish is not introduced (that is, if $y(t) = 0$ for all t), then the system reduces to $dx/dt = -0.2x$. In this case, we have an exponential decay model as in Section 1.1, and the native fish population tends to its equilibrium level. (Remember: the quantity $x(t)$ is the difference between the native fish population and its equilibrium level, not the actual of fish population.) Thus, the model agrees with the system as described.

(c) There are two lines consisting of straight-line solutions, and the solutions with initial conditions on these lines are asymptotic to the origin as $t \to 0$. To find these lines, we must compute the eigenvalues and eigenvectors.

The characteristic polynomial of the given matrix is

$$(-0.2 - \lambda)(-0.1 - \lambda),$$

and hence the eigenvalues are $\lambda_1 = -0.2$ and $\lambda_2 = -0.1$.
To find an eigenvector for $\lambda_1 = -0.2$, we must solve

$$\begin{pmatrix} -0.2 & -0.1 \\ 0.0 & -0.1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -0.2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

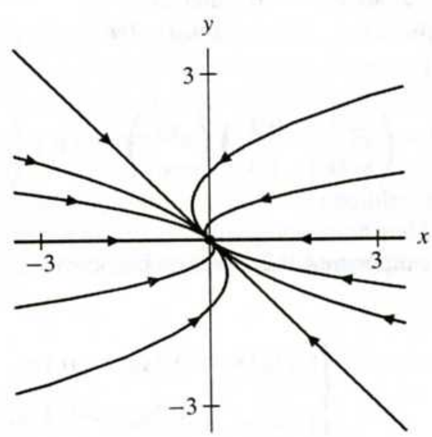
Rewritten in terms of components, this equation becomes

$$\begin{cases} -0.2x_0 - 0.1y_0 = -0.2x_0 \\ -0.1y_0 = -0.2y_0, \end{cases}$$

which is equivalent to

$$\begin{cases} -0.1y_0 = 0 \\ 0.1y_0 = 0. \end{cases}$$

If we multiply the second equation by -1, we obtain the first equation. Therefore, the equations are redundant and any vector (x_0, y_0) that satisfies the first equation is an eigenvector. Setting $x_0 = 1$ yields the eigenvector $\mathbf{V}_1 = (1, 0)$.
To find an eigenvector for $\lambda_2 = -0.1$, we repeat the process with λ_2 in replace of λ_1, and we obtain the eigenvector $\mathbf{V}_2 = (1, -1)$
Since both eigenvalues are negative, solutions with initial conditions that lie on the lines through the origin determined by the eigenvectors (the x-axis and the line $y = -x$) tend toward the origin.



(d) Using the phase portrait shown in part (c), we see that solutions with initial conditions of the form $(0, y)$, $y > 0$, move through the second quadrant and tend toward the equilibrium point at the origin. Thus, our model predicts that, if a small number of new fish are added to the lake, the native population drops below its equilibrium level since x is negative. The new fish will die out and the native fish will return to equilibrium.