## 11.6 Directional Derivatives and the Gradient Vector

1. First we draw a line passing through Raleigh and the eye of the hurricane. We can approximate the directional derivative at Raleigh in the direction of the eye of the hurricane by the average rate of change of pressure between the points where this line intersects the contour lines closest to Raleigh. In the direction of the eye of the hurricane, the pressure changes from 996 millibars to 992 millibars. We estimate the distance between these two points to be approximately 40 miles, so the rate of change of pressure in the direction given is approximately $\frac{992-996}{40} = -0.1$ millibar/mi.

2. First we draw a line passing through Muskegon and Ludington. We approximate the directional derivative at Muskegon in the direction of Ludington by the average rate of change of snowfall between the points where the line intersects the contour lines closest to Muskegon. In the direction of Ludington, the snowfall changes from 60 to 70 inches. We estimate the distance between these two points to be approximately 28 miles, so the rate of change of annual snowfall in the direction given is approximately $\frac{70-60}{28} \approx 0.36$ in/mi. [If we talk of snowfall (rather than annual snowfall), the units are (in/year)/mi.]

3. $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ is a unit vector, so by Theorem 3, $D_{\mathbf{u}} f(16,30) = f_T(16,30) \cdot \frac{1}{\sqrt{2}} + f_v(16,30) \cdot \frac{1}{\sqrt{2}}$. We can estimate the value of $f_T(16,30)$ following the technique used in Exercise 11.3.3.
$f_T(16,30) = \lim\limits_{h\to 0} \dfrac{f(16+h,30) - f(16,30)}{h}$, and taking $h = 4$ and $h = -4$ we have
$f_T(16,30) \approx \dfrac{f(20,30) - f(16,30)}{4} = \dfrac{14-9}{4} = 1.25$, $f_T(16,30) \approx \dfrac{f(12,30) - f(16,30)}{-4} = \dfrac{3-9}{-4} = 1.5$.
Averaging these values, we estimate $f_T(16,30) \approx 1.375$. Similarly,
$f_v(16,30) = \lim\limits_{h\to 0} \dfrac{f(16,30+h) - f(16,30)}{h}$, and taking $h = 10$ and $h = -10$ gives
$f_v(16,30) \approx \dfrac{f(16,40) - f(16,30)}{10} = \dfrac{7-9}{10} = -0.2$, $f_v(16,30) \approx \dfrac{f(16,20) - f(16,30)}{-10} = \dfrac{11-9}{-10} = -0.2$.
Averaging these values, we estimate $f_v(16,30) \approx -0.2$. Thus
$D_{\mathbf{u}} f(16,30) = f_T(16,30) \cdot \frac{1}{\sqrt{2}} + f_v(16,30) \cdot \frac{1}{\sqrt{2}} \approx (1.375)\frac{1}{\sqrt{2}} + (-0.2)\frac{1}{\sqrt{2}} \approx 0.83$.

4. $f(x,y) = \sin(x+2y) \;\Rightarrow\; f_x(x,y) = \cos(x+2y)$ and $f_y(x,y) = 2\cos(x+2y)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{3\pi}{4}$, then from Equation 6,
$D_{\mathbf{u}} f(4,-2) = f_x(4,-2)\cos\frac{3\pi}{4} + f_y(4,-2)\sin\frac{3\pi}{4} = (\cos 0)\left(-\frac{\sqrt{2}}{2}\right) + 2(\cos 0)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$.

5. $f(x,y) = \sqrt{5x-4y} \;\Rightarrow\; f_x(x,y) = \frac{1}{2}(5x-4y)^{-1/2}(5) = \dfrac{5}{2\sqrt{5x-4y}}$ and
$f_y(x,y) = \frac{1}{2}(5x-4y)^{-1/2}(-4) = -\dfrac{2}{\sqrt{5x-4y}}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = -\frac{\pi}{6}$, then from Equation 6, $D_{\mathbf{u}} f(4,1) = f_x(4,1)\cos\left(-\frac{\pi}{6}\right) + f_y(4,1)\sin\left(-\frac{\pi}{6}\right) = \frac{5}{8}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{5\sqrt{3}}{16} + \frac{1}{4}$.

6. $f(x,y) = xe^{-2y} \;\Rightarrow\; f_x(x,y) = e^{-2y}$ and $f_y(x,y) = -2xe^{-2y}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{2}$, then $D_{\mathbf{u}} f(5,0) = f_x(5,0)\cos\frac{\pi}{2} + f_y(5,0)\sin\frac{\pi}{2} = 1\cdot 0 + (-10)1 = -10$.

**7.** $f(x,y) = 5xy^2 - 4x^3y$

(a) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 5y^2 - 12x^2y, 10xy - 4x^3\rangle$

(b) $\nabla f(1,2) = \langle 5(2)^2 - 12(1)^2(2), 10(1)(2) - 4(1)^3\rangle = \langle -4, 16\rangle$

(c) By Equation 9, $D_{\mathbf{u}}f(1,2) = \nabla f(1,2)\cdot\mathbf{u} = \langle -4,16\rangle\cdot\left\langle \frac{5}{13},\frac{12}{13}\right\rangle = (-4)\left(\frac{5}{13}\right) + (16)\left(\frac{12}{13}\right) = \frac{172}{13}$.

**8.** $f(x,y) = y\ln x$

(a) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle y/x, \ln x\rangle$ (b) $\nabla f(1,-3) = \left\langle \frac{-3}{1}, \ln 1\right\rangle = \langle -3, 0\rangle$

(c) By Equation 9, $D_{\mathbf{u}}f(1,-3) = \nabla f(1,-3)\cdot\mathbf{u} = \langle -3,0\rangle\cdot\left\langle -\frac{4}{5},\frac{3}{5}\right\rangle = \frac{12}{5}$.

**9.** $f(x,y,z) = xy^2z^3$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$

(b) $\nabla f(1,-2,1) = \langle 4,-4,12\rangle$

(c) $\nabla f(1,-2,1)\cdot\mathbf{u} = \frac{4}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{12}{\sqrt{3}} = \frac{20}{\sqrt{3}}$

**10.** $f(x,y,z) = xy + yz^2 + xz^3$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle y + z^3, x + z^2, 2yz + 3xz^2\rangle$

(b) $\nabla f(2,0,3) = \langle 27, 11, 54\rangle$

(c) $\nabla f(2,0,3)\cdot\mathbf{u} = \frac{1}{3}(-54 - 11 + 108) = \frac{43}{3}$

**11.** $f(x,y) = 1 + 2x\sqrt{y} \Rightarrow \nabla f(x,y) = \left\langle 2\sqrt{y}, 2x\cdot\frac{1}{2}y^{-1/2}\right\rangle = \langle 2\sqrt{y}, x/\sqrt{y}\rangle$, $\nabla f(3,4) = \left\langle 4, \frac{3}{2}\right\rangle$,
and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2+(-3)^2}}\langle 4,-3\rangle = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$, so
$D_{\mathbf{u}}f(3,4) = \nabla f(3,4)\cdot\mathbf{u} = \left\langle 4,\frac{3}{2}\right\rangle\cdot\left\langle \frac{4}{5},-\frac{3}{5}\right\rangle = \frac{23}{10}$

**12.** $g(r,\theta) = e^{-r}\sin\theta \Rightarrow \nabla g(r,\theta) = (-e^{-r}\sin\theta)\,\mathbf{i} + (e^{-r}\cos\theta)\,\mathbf{j}$, $\nabla g\left(0,\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$,
and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$, so
$D_{\mathbf{u}}g\left(0,\frac{\pi}{3}\right) = \nabla g\left(0,\frac{\pi}{3}\right)\cdot\mathbf{u} = \left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)\cdot\frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j}) = -\frac{3\sqrt{3}}{2\sqrt{13}} - \frac{1}{\sqrt{13}} = -\frac{3\sqrt{3}+2}{2\sqrt{13}}$.

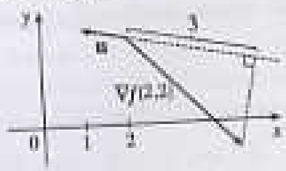
**13.** $f(x,y,z) = \sqrt{x^2+y^2+z^2} \Rightarrow \nabla f(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle$,
$\nabla f(1,2,-2) = \left\langle \frac{1}{3},\frac{2}{3},-\frac{2}{3}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{9}\langle -6,6,-3\rangle = \left\langle -\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right\rangle$, so
$D_{\mathbf{u}}f(1,2,-2) = \nabla f(1,2,-2)\cdot\mathbf{u} = \left\langle \frac{1}{3},\frac{2}{3},-\frac{2}{3}\right\rangle\cdot\left\langle -\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right\rangle = \frac{4}{9}$.

**14.** $f(x,y,z) = \frac{x}{y+z} \Rightarrow \nabla f(x,y,z) = \left\langle \frac{1}{y+z}, -\frac{x}{(y+z)^2}, -\frac{x}{(y+z)^2}\right\rangle$,
$\nabla f(4,1,1) = \left\langle \frac{1}{2},-1,-1\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle$, so
$D_{\mathbf{u}}f(4,1,1) = \nabla f(4,1,1)\cdot\mathbf{u} = \left\langle \frac{1}{2},-1,-1\right\rangle\cdot\frac{1}{\sqrt{14}}\langle 1,2,3\rangle = -\frac{9}{2\sqrt{14}}$.

**15.** $g(x,y,z) = x\tan^{-1}(y/z) \Rightarrow \nabla g(x,y,z) = \langle \tan^{-1}(y/z), xz/(y^2+z^2), -xy/(y^2+z^2)\rangle$,
$\nabla g(1,2,-2) = \left\langle -\frac{\pi}{4}, -\frac{1}{4}, -\frac{1}{4}\right\rangle$, $\mathbf{u} = \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle$ and

$$D_{\mathbf{u}}g(1,2,-2) = \frac{(-\pi)(1)}{4\sqrt{3}} + \frac{(-1)(1)}{4\sqrt{3}} + \frac{(-1)(-1)}{4\sqrt{3}} = -\frac{\pi}{4\sqrt{3}}.$$

**16.** $D_{\mathbf{u}} f(2,2) = \nabla f(2,2) \cdot \mathbf{u}$, the scalar projection of $\nabla f(2,2)$ onto $\mathbf{u}$, so we draw a perpendicular from the tip of $\nabla f(2,2)$ to the line containing $\mathbf{u}$. We can use the point $(2,2)$ to determine the scale of the axes, and we estimate the length of the projection to be approximately 3.0 units. Since the angle between $\nabla f(2,2)$ and $\mathbf{u}$ is greater than $90^\circ$, the scalar projection is negative. Thus $D_{\mathbf{u}} f(2,2) \approx -3$.




**17.** $f(x,y) = \sqrt{xy} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle$, so $\nabla f(2,8) = \left\langle 1, \frac{1}{4} \right\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8 \rangle = \langle 3, -4 \rangle$ is $\mathbf{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$, so $D_{\mathbf{u}} f(2,8) = \nabla f(2,8) \cdot \mathbf{u} = \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{2}{5}$.

**18.** $f(x,y,z) = x^2 + y^2 + z^2 \quad \Rightarrow \quad \nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$, so $\nabla f(2,1,3) = \langle 4, 2, 6 \rangle$. The unit vector in the direction of $\overrightarrow{PO} = \langle -2, -1, -3 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{14}} \langle -2, -1, -3 \rangle$, so $D_{\mathbf{u}} f(2,1,3) = \nabla f(2,1,3) \cdot \mathbf{u} = \langle 4, 2, 6 \rangle \cdot \frac{1}{\sqrt{14}} \langle -2, -1, -3 \rangle = -\frac{28}{\sqrt{14}} = -2\sqrt{14}$.

**19.** $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $\nabla f(1,0) = \langle 0, 1 \rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0, 1 \rangle$.

**20.** $f(x,y) = \ln(x^2 + y^2) \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$, $\nabla f(1,2) = \left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$. Thus the maximum rate of change is $|\nabla f(1,2)| = \frac{2\sqrt{5}}{5}$ in the direction $\left\langle \frac{2}{5}, \frac{4}{5} \right\rangle$ or $\langle 2, 4 \rangle$.

**21.** $f(x,y,z) = x + y/z \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle 1, \frac{1}{z}, -\frac{y}{z^2} \right\rangle$, so the maximum rate of change is $|\nabla f(4,3,-1)| = \sqrt{11}$ in the direction $\langle 1, -1, -3 \rangle$.

**22.** $f(x,y,z) = x^2y^3z^4 \quad \Rightarrow \quad \nabla f(x,y,z) = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle$, $\nabla f(1,1,1) = \langle 2, 3, 4 \rangle$. Thus the maximum rate of change is $|\nabla f(1,1,1)| = \sqrt{29}$ in the direction $\langle 2, 3, 4 \rangle$.

**23.** (a) As in the proof of Theorem 15, $D_{\mathbf{u}} f = |\nabla f| \cos\theta$. Since the minimum value of $\cos\theta$ is $-1$ occurring when $\theta = \pi$, the minimum value of $D_{\mathbf{u}} f$ is $-|\nabla f|$ occurring when $\theta = \pi$, that is when $\mathbf{u}$ is in the opposite direction of $\nabla f$ (assuming $\nabla f \neq 0$).

(b) $f(x,y) = x^4y - x^2y^3 \quad \Rightarrow \quad \nabla f(x,y) = \langle 4x^3y - 2xy^3, x^4 - 3x^2y^2 \rangle$, so $f$ decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$.

**24.** $f(x,y) = x^2 + \sin xy \quad \Rightarrow \quad f_x(x,y) = 2x + y\cos xy$, $f_y(x,y) = x\cos xy$ and $f_x(1,0) = 2(1) + (0)\cos 0 = 2$, $f_y(1,0) = (1)\cos 0 = 1$. If $\mathbf{u}$ is a unit vector which makes an angle $\theta$ with the positive $x$-axis, then $D_{\mathbf{u}} f(1,0) = f_x(1,0)\cos\theta + f_y(1,0)\sin\theta = 2\cos\theta + \sin\theta$. We want $D_{\mathbf{u}} f(1,0) = 1$, so $2\cos\theta + \sin\theta = 1 \quad \Rightarrow \quad \sin\theta = 1 - 2\cos\theta \quad \Rightarrow \quad \sin^2\theta = (1 - 2\cos\theta)^2 \quad \Rightarrow$ $1 - \cos^2\theta = 1 - 4\cos\theta + 4\cos^2\theta \quad \Rightarrow \quad 5\cos^2\theta - 4\cos\theta = 0 \quad \Rightarrow \quad \cos\theta(5\cos\theta - 4) = 0 \quad \Rightarrow \quad \cos\theta = 0$ or $\cos\theta = \frac{4}{5} \quad \Rightarrow \quad \theta = \frac{\pi}{2}$ or $\theta = 2\pi - \cos^{-1}\left(\frac{4}{5}\right) \approx 5.64$.

**25.** The direction of fastest change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points $(x,y)$ where $\nabla f(x,y)$ is parallel to $\mathbf{i} + \mathbf{j} \quad \Leftrightarrow \quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = k(\mathbf{i} + \mathbf{j}) \quad \Leftrightarrow \quad k = 2x - 2$ and $k = 2y - 4$. Then ... $y = x + 1$, so the direction of fastest change is $\mathbf{i} + \mathbf{j}$ at all points on the line $y = x + 1$.

**26.** The fisherman is traveling in the direction $\langle -80, -60\rangle$. A unit vector in this direction is $\mathbf{u} = \frac{1}{100}\langle -80, -60\rangle = \langle -\frac{4}{5}, -\frac{3}{5}\rangle$, and if the depth of the lake is given by $f(x, y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x, y) = \langle 0.04x, -0.003y^2\rangle$. $D_{\mathbf{u}} f(80, 60) = \nabla f(80, 60) \cdot \mathbf{u} = \langle 3.2, -10.8\rangle \cdot \langle -\frac{4}{5}, -\frac{3}{5}\rangle = 3.92$. Since $D_{\mathbf{u}} f(80, 60)$ is positive, the depth of the lake is increasing near $(80, 60)$ in the direction toward the buoy.

**27.** $T = \dfrac{k}{\sqrt{x^2 + y^2 + z^2}}$ and $120 = T(1, 2, 2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1, -1, 1\rangle}{\sqrt{3}}$,

$$D_{\mathbf{u}} T(1, 2, 2) = \nabla T(1, 2, 2) \cdot \mathbf{u} = \left[-360(x^2 + y^2 + z^2)^{-3/2}\langle x, y, z\rangle\right]_{(1,2,2)} \cdot \mathbf{u}$$
$$= -\tfrac{40}{3}\langle 1, 2, 2\rangle \cdot \tfrac{1}{\sqrt{3}}\langle 1, -1, 1\rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360(x^2 + y^2 + z^2)^{-3/2}\langle x, y, z\rangle$, and since $\langle x, y, z\rangle$ is the position vector of the point $(x, y, z)$, the vector $-\langle x, y, z\rangle$, and thus $\nabla T$, always points toward the origin.

**28.** $\nabla T = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z\rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1, -2, 1\rangle$, $\nabla T(2, -1, 2) = -400e^{-43}\langle 2, -3, 18\rangle$ and

$D_{\mathbf{u}} T(2, -1, 2) = \left(-\dfrac{400e^{-43}}{\sqrt{6}}\right)(26) = -\dfrac{5200\sqrt{6}}{3e^{43}}$ °C/m.

(b) $\nabla T(2, -1, 2) = 400e^{-43}\langle -2, 3, -18\rangle$ or equivalently $\langle -2, 3, -18\rangle$.

(c) $|\nabla T| = 400e^{-x^2-3y^2-9z^2}\sqrt{x^2 + 9y^2 + 81z^2}$ °C/m is the maximum rate of increase. At $(2, -1, 2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}$ °C/m.

**29.** $\nabla V(x, y, z) = \langle 10x - 3y + yz, xz - 3x, xy\rangle$, $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$

(a) $D_{\mathbf{u}} V(3, 4, 5) = \langle 38, 6, 12\rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3, 4, 5) = \langle 38, 6, 12\rangle$ or equivalently $\langle 19, 3, 6\rangle$.

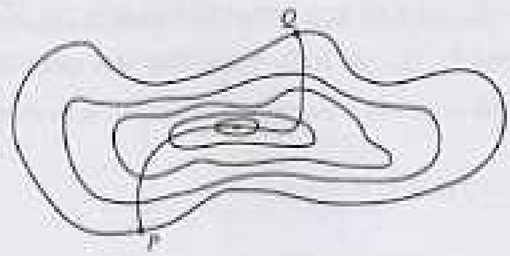
(c) $|\nabla V(3, 4, 5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624} = 2\sqrt{406}$

**30.** (a) Let $z = f(x, y) = 1000 - 0.01x^2 - 0.02y^2$. Then $\nabla f(x, y) = \langle -0.02x, -0.04y\rangle$. Proceed in the direction $\nabla f(60, 100) = \langle -1.2, -4\rangle$.

(b) The maximum slope is equal to the maximum directional derivative, which is $|\langle -1.2, -4\rangle| = \sqrt{17.44}$ and $\theta = \tan^{-1}\sqrt{17.44} \approx 76.5°$.

**31.** A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}} f(1, 3) = f_x(1, 3) = 3$ and $D_{\overrightarrow{AC}} f(1, 3) = f_y(1, 3) = 26$. Therefore $\nabla f(1, 3) = \langle f_x(1, 3), f_y(1, 3)\rangle = \langle 3, 26\rangle$, and by definition, $D_{\overrightarrow{AD}} f(1, 3) = \nabla f \cdot \mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\langle \frac{5}{13}, \frac{12}{13}\rangle$. Therefore, $D_{\overrightarrow{AD}} f(1, 3) = \langle 3, 26\rangle \cdot \langle \frac{5}{13}, \frac{12}{13}\rangle = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}$.

**32.** The curve of steepest ascent is perpendicular to all of the contour lines.




**33.** (a) $\nabla(au+bv) = \left\langle \frac{\partial(au+bv)}{\partial x}, \frac{\partial(au+bv)}{\partial y} \right\rangle = \left\langle a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x}, a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y} \right\rangle$

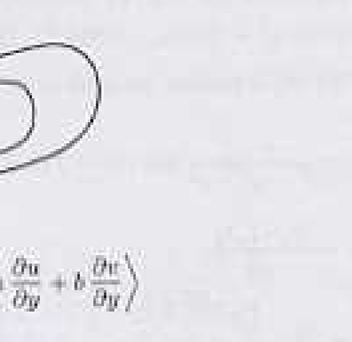
$$= a\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle + b\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = a\,\nabla u + b\,\nabla v$$

(b) $\nabla(uv) = \left\langle v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}, v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} \right\rangle = v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle + u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = v\,\nabla u + u\,\nabla v$

(c) $\nabla\left(\frac{u}{v}\right) = \left\langle \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}, \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2} \right\rangle = \frac{v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle - u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle}{v^2} = \frac{v\,\nabla u - u\,\nabla v}{v^2}$

(d) $\nabla u^n = \left\langle \frac{\partial(u^n)}{\partial x}, \frac{\partial(u^n)}{\partial y} \right\rangle = \left\langle nu^{n-1}\frac{\partial u}{\partial x}, nu^{n-1}\frac{\partial u}{\partial y} \right\rangle = nu^{n-1}\,\nabla u$

**34.** If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of $f$ that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of $f$ at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to $-2$ and $-3$ with an estimated distance of 0.5 units. Thus the rate of change is approximately $\frac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.




**35.** Let $F(x,y,z) = x^2 + 2y^2 + 3z^2$. Then $x^2 + 2y^2 + 3z^2 = 21$ is a level surface of $F$.
$F_x(x,y,z) = 2x \;\Rightarrow\; F_x(4,-1,1) = 8$, $F_y(x,y,z) = 4y \;\Rightarrow\; F_y(4,-1,1) = -4$, and
$F_z(x,y,z) = 6z \;\Rightarrow\; F_z(4,-1,1) = 6$.

(a) Equation 19 gives an equation of the tangent plane at $(4,-1,1)$ as $8(x-4) - 4[y-(-1)] + 6(z-1) = 0$ or $4x - 2y + 3z = 21$.

(b) By Equation 20, the normal line has symmetric equations $\frac{x-4}{8} = \frac{y+1}{-4} = \frac{z-1}{6}$ or $\frac{x-4}{4} = \frac{y+1}{-2} = \frac{z-1}{3}$.