Hence, we have

$$\mathbf{Y}_{\text{re}}(t) = e^t \begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\text{im}}(t) = e^t \begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

2. We use Euler's formula to write

$$\begin{aligned} e^{(2+i)t} \begin{pmatrix} 1 \\ 4i \end{pmatrix} &= e^{2t} e^{it} \begin{pmatrix} 1 \\ 4i \end{pmatrix} \\ &= e^{2t}(\cos t + i \sin t) \begin{pmatrix} 1 \\ 4i \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} \cos t \\ -4\sin t \end{pmatrix} + e^{2t} \begin{pmatrix} \sin t \\ 4\cos t \end{pmatrix} \end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\text{re}}(t) = e^{2t} \begin{pmatrix} \cos t \\ -4\sin t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\text{im}}(t) = e^{2t} \begin{pmatrix} \sin t \\ 4\cos t \end{pmatrix}.$$

3. **(a)** The characteristic equation is

$$(-\lambda)^2 + 4 = \lambda^2 + 4 = 0,$$

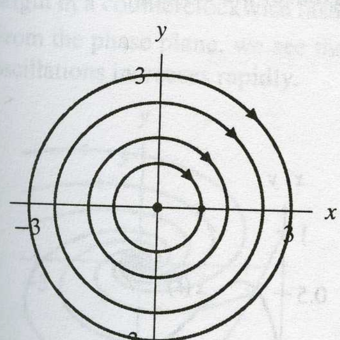
and the eigenvalues are $\lambda = \pm 2i$.

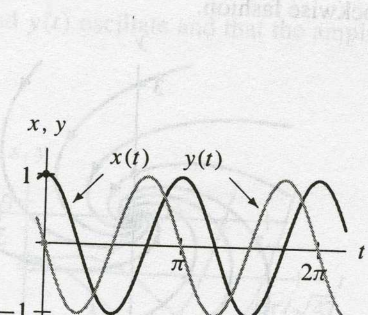
(b) Since the real part of the eigenvalues are 0, the origin is a center.

(c) Since $\lambda = \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At $(1, 0)$, the tangent vector is $(-2, 0)$. Therefore, the direction of oscillation is clockwise.

(e) According to the phase plane, $x(t)$ and $y(t)$ are periodic with period π. At the initial condition $(1, 0)$, both $x(t)$ and $y(t)$ are initially decreasing.





4. **(a)** The characteristic equation is

$$(2-\lambda)(6-\lambda)+8=\lambda^2-8\lambda+20,$$

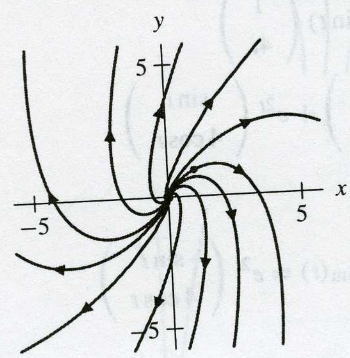
and the eigenvalues are $\lambda = 4 \pm 2i$.

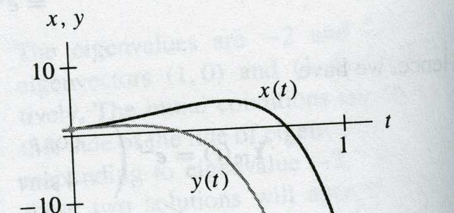
(b) Since the real part of the eigenvalues is positive, the origin is a spiral source.

(c) Since $\lambda = 4 \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At the point $(1, 0)$, the tangent vector is $(2, -4)$. Therefore, the solution curves spiral around the origin in a clockwise fashion.

(e) Since $d\mathbf{Y}/dt = (4, 2)$ at $\mathbf{Y}_0 = (1, 1)$, both $x(t)$ and $y(t)$ increase initially. The distance between successive zeros is π, and the amplitudes of both $x(t)$ and $y(t)$ are increasing.





5. **(a)** The characteristic polynomial is

$$(-1-\lambda)(-1-\lambda)+2=\lambda^2+2\lambda+3,$$

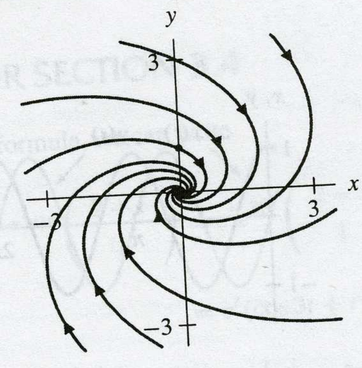
so the eigenvalues are $\lambda = -1 \pm i\sqrt{2}$.

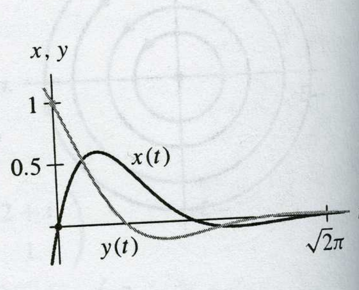
(b) The eigenvalues are complex and the real part is negative so the origin is a spiral sink.

(c) The natural period is $2\pi/\sqrt{2} = \sqrt{2}\pi$. The natural frequency is $1/(\sqrt{2}\pi)$.

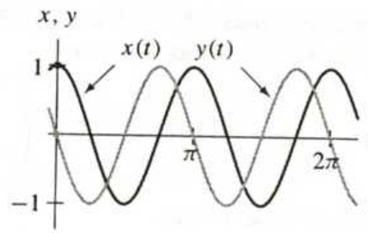
(d) At the point $(1, 0)$, the vector field is $(-1, -1)$. Hence, the solution curves must spiral in a clockwise fashion.

(e)





(c)



10. **(a)** According to Exercise 4, the eigenvalues are $\lambda = 4 \pm 2i$. The eigenvectors (x, y) associated to the eigenvalue $4 + 2i$ must satisfy the equation $y = (1 + i)x$. Hence, using the eigenvector $(1, 1 + i)$, we obtain the complex-valued solution

$$\mathbf{Y}(t) = e^{(4+2i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{4t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + i e^{4t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

From the real and imaginary parts of $\mathbf{Y}(t)$, we obtain the general solution

$$\mathbf{Y}(t) = k_1 e^{4t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + k_2 e^{4t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

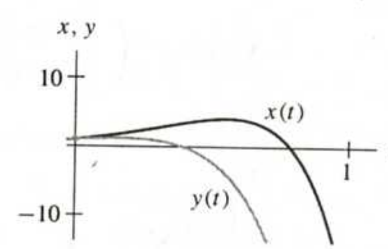
(b) Using the initial condition, we have

$$k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and thus $k_1 = 1$ and $k_2 = 0$. The desired solution is

$$\mathbf{Y}(t) = e^{4t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

(c)



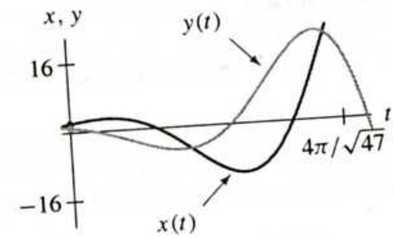
11. **(a)** To find the general solution, we find the eigenvectors from the characteristic polynomial

$$(-1 - \lambda)(-1 - \lambda) + 2 = \lambda^2 + 2\lambda + 3.$$

The eigenvalues are $\lambda = -1 \pm i\sqrt{2}$. To find an eigenvector associated to the eigenvector $-1 + i\sqrt{2}$, we must solve the equations

$$\begin{cases} -x + 2y = \left(-1 + i\sqrt{2}\right)x \\ -x - y = \left(-1 + i\sqrt{2}\right)y. \end{cases}$$

(c)



15. **(a)** In the case of complex eigenvalues, the function $x(t)$ oscillates about $x = 0$ with constant period, and the amplitude of successive oscillations is either increasing, decreasing, or constant depending on the sign of the real part of the eigenvalue. The graphs that satisfy these properties are (ii) and (v).

(b) For (ii), the natural period is about 1.5 and, since the amplitude tends toward zero as t increases, the origin is a sink. For (v), the natural period is about 1.25 and, since the amplitude increases as t increases, the origin is a source.

(c) (i) The time between successive zeros is not constant.
(iii) The amplitude is not monotonically decreasing or increasing.
(iv) Oscillation stops at some t.
(vi) Oscillation starts at some t. There was no prior oscillation.

16. The characteristic polynomial is

$$(a-\lambda)(a-\lambda)+b^2 = \lambda^2 - 2a\lambda + (a^2+b^2),$$

so the eigenvalues are

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = a \pm \frac{-4b^2}{2} = a \pm \sqrt{-b^2}.$$

Since $b^2 > 0$, the eigenvalues must be complex. In fact, they equal $a \pm i|b|$.

17. We know that $\lambda_1 = \alpha + i\beta$ satisfies the equation $\lambda_1^2 + a\lambda_1 + b = 0$. Therefore, if we take the complex conjugate all of the terms in this equation, we obtain

$$(\alpha - i\beta)^2 + a(\alpha - i\beta) + b = 0,$$

since a and b are real. The complex conjugate of λ_1 is $\lambda_2 = \alpha - i\beta$, and we have

$$\lambda_2^2 + a\lambda_2 + b = 0.$$

Therefore, λ_2 is also a root.

18. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If (x_0, y_0) is an eigenvector associated to the eigenvalue $\alpha + i\beta$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} (\alpha+i\beta)x_0 \\ (\alpha+i\beta)y_0 \end{pmatrix}.$$

Then $ax_0 + by_0 = (\alpha + i\beta)x_0$, which is equivalent to

$$y_0 = \frac{a - \alpha + i\beta}{b} x_0.$$

Suppose x_0 is real and nonzero, then the imaginary part of y_0 is $\beta x_0/b$. Since $\beta \neq 0$, the imaginary part of y_0 must be nonzero. (Note: If $b = 0$, then the eigenvalues are a and d. In other words, they are not complex, so the hypothesis of the exercise is not satisfied).

19. Suppose $\mathbf{Y}_2 = k\mathbf{Y}_1$ for some constant k. Then, $\mathbf{Y}_0 = (1 + ik)\mathbf{Y}_1$. Since $\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$, we have

$$(1 + ik)\mathbf{A}\mathbf{Y}_1 = \lambda(1 + ik)\mathbf{Y}_1.$$

Thus, $\mathbf{A}\mathbf{Y}_1 = \lambda\mathbf{Y}_1$. Now note that the left-hand side, $\mathbf{A}\mathbf{Y}_1$, is a real vector. However, since λ is complex and $\mathbf{Y}_1$ is real, the right-hand side is complex (that is, it has a nonzero imaginary part). Thus, we have a contradiction, and $\mathbf{Y}_1$ and $\mathbf{Y}_2$ must be linearly independent.

20. If $\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$, then we can take complex conjugates of both sides to obtain $\overline{\mathbf{A}\mathbf{Y}_0} = \overline{\lambda\mathbf{Y}_0}$ (where the complex conjugate of a vector or matrix is the complex conjugate of its entries). But $\overline{\mathbf{A}\mathbf{Y}_0} = \overline{\mathbf{A}}\,\overline{\mathbf{Y}_0} = \mathbf{A}\overline{\mathbf{Y}_0}$ because $\mathbf{A}$ is real. Also, $\overline{\lambda\mathbf{Y}_0} = \overline{\lambda}\,\overline{\mathbf{Y}_0}$. Hence, $\mathbf{A}\overline{\mathbf{Y}_0} = \overline{\lambda}\overline{\mathbf{Y}_0}$. In other words, $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}$ with eigenvector $\overline{\mathbf{Y}_0}$.

21. **(a)** The factor $e^{-\alpha t}$ is positive for all t. Hence, the zeros of $x(t)$ are exactly the zeros of $\sin\beta t$. Suppose t_1 and t_2 are successive zeros (that is, $t_1 < t_2$, $x(t_1) = x(t_2) = 0$, and $x(t) \neq 0$ for $t_1 < t < t_2$), then $\beta t_2 - \beta t_1 = \pi$. In other words, $t_2 - t_1 = \pi/\beta$.

(b) By the nature of sine function, local maxima and local minima appear alternately. Therefore, we look for t_1 and t_2 such that $x'(t_1) = x'(t_2) = 0$ and $x'(t) \neq 0$ for $t_1 < t < t_2$. From

$$x'(t) = e^{-\alpha t}(-\alpha \sin\beta t + \beta\cos\beta t) = 0,$$

we know that $\tan\beta t = \beta/\alpha$ if t is corresponds to a local extremum. Since the tangent function is periodic with period π, $\beta(t_2 - t_1) = \pi$. Hence, $t_2 - t_1 = \pi/\beta$. Note that the distance between a local minimum and the following local maximum of $x(t)$ is constant over t.

(c) From part (b), we know that the distance between the first local maximum and the first local minimum is π/β and the distance between the first local minimum and the second local maximum is π/β. Therefore, the distance between the first two local maxima of $x(t)$ is $2\pi/\beta$.

(d) From part (b), we know that the first local maximum of $x(t)$ occurs at $t = (\arctan(\beta/\alpha))/\beta$.

22. Consider the point in the plane determined by the coordinates (k_1, k_2), and let ϕ be an angle such that $K\cos\phi = k_1$ and $K\sin\phi = k_2$. (Such an angle exists since $(K\cos\phi, K\sin\phi)$ parameterizes the circle through (k_1, k_2) centered at the origin. In fact, there are infinitely many such ϕ, all differing by integer multiples of 2π.) Then

$$\begin{aligned} x(t) &= k_1\cos\beta t + k_2\sin\beta t \\ &= K\cos\phi\cos\beta t + K\sin\phi\sin\beta t \\ &= K\cos(\beta t - \phi). \end{aligned}$$

The last equality comes from the trigonometric identity for the cosine of the difference of two angles.

23. **(a)** The corresponding first-order system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

(b) The characteristic polynomial is

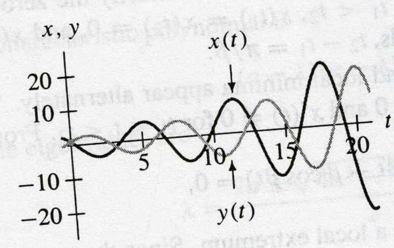
$$(-\lambda)(-p - \lambda) + q = \lambda^2 + p\lambda + q,$$

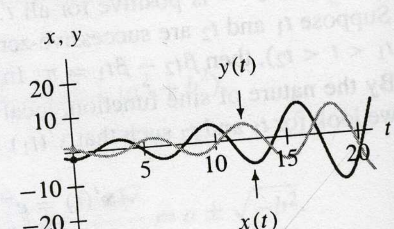
so the eigenvalues are $\lambda = (-p \pm \sqrt{p^2 - 4q})/2$. Hence, the eigenvalues are complex if and only if $p^2 < 4q$.

(c) In order to have a spiral sink, we must have $p^2 < 4q$ (to make the eigenvalues complex) and $p > 0$ (to make the real part of the eigenvalues negative). To have a center, we must have $p = 0$.

(d) The vector field at $(1, 0)$ is $(0, -q)$. Hence, if $q > 0$, then the vector field points down along the entire y-axis, and the solution curves spiral about the origin in a clockwise fashion.

24. Note that the graphs have the same period and exponential rate of growth.





25. There is no spiral saddle because a linear saddle is a linear system where some solutions approach the origin and some move away. If one solution spirals toward (or away from) the origin, then we can multiply that solution by any constant, scaling it so that it goes through any point in the plane. This scaled solution is still a solution of the system (recall the Linearity Principle), so every solution spirals in the same way, either toward or away from the origin.

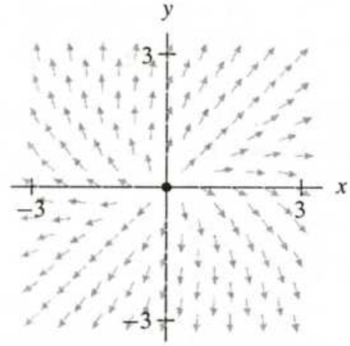
26. The eigenvalues are $\pm i$. Using the usual method to find eigenvectors, we see that the eigenvectors corresponding to the eigenvalue i satisfy the equation $10y = (3 + i)x$. We use the eigenvector $\mathbf{V}_0 = (10, 3 + i)$ to determine the general solution. It is

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} 10 \cos t \\ 3 \cos t - \sin t \end{pmatrix} + k_2 \begin{pmatrix} 10 \sin t \\ \cos t + 3 \sin t \end{pmatrix}.$$

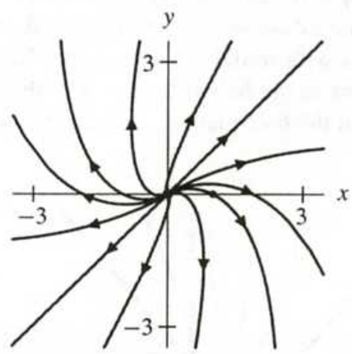
In terms of components, we have

$$x(t) = 10k_1 \cos t + 10k_2 \sin t$$
$$y(t) = (3k_1 + k_2) \cos t + (3k_2 - k_1) \sin t.$$

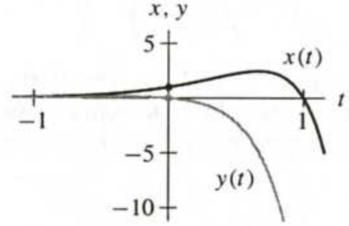
(c) Note the straight-line solutions along the line $y = x$.



(d) Since the sole eigenvalue is positive, all solutions except the equilibrium solution are unbounded as t increases. As $t \to -\infty$, the solutions with initial conditions in the half-plane $y > x$ tend to the origin tangent to the half-line $y = x$ with $y < 0$. Similarly, solutions with initial conditions in the half-plane $y < x$ tend to the origin tangent to the half-line $y = x$ with $y > 0$. Note the solution curve that goes through the initial condition $(1, 0)$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (2, -1)$. Hence, $x(t)$ is initially increasing, and $y(t)$ is initially decreasing.



3. **(a)** The characteristic equation is

$$(-2 - \lambda)(-4 - \lambda) + 1 = (\lambda + 3)^2 = 0,$$

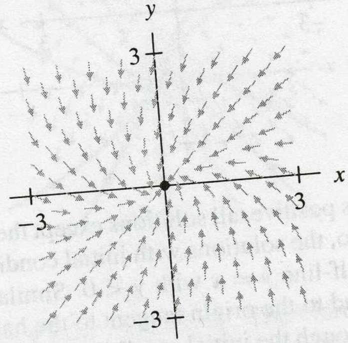
and the eigenvalue is $\lambda = -3$.

(b) To find an eigenvector, we solve the simultaneous equations

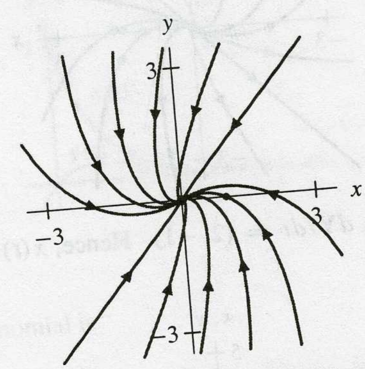
$$\begin{cases} -2x - y = -3x \\ x - 4y = -3y. \end{cases}$$

Then, $y = x$, and one eigenvector is $(1, 1)$.

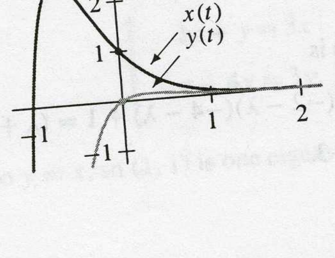
(c) Note the straight-line solutions along the line $y = x$.



(d) Since the eigenvalue is negative, any solution on the line $y = x$ tends toward the origin along $y = x$ as t increases. According to the direction field, every solution tends to the origin as t increases. The solutions with initial conditions that lie in the half-plane $y > x$ eventually approach the origin tangent to the half-line $y = x$ with $y < 0$. Similarly, the solutions with initial conditions that lie in the half-plane $y < x$ eventually approach the origin tangent to the line $y = x$ with $y > 0$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (-2, 1)$. Therefore, $x(t)$ initially decreases and $y(t)$ initially increases. The solution eventually approaches the origin tangent to the line $y = x$. Since the solution curve never crosses the line $y = x$, the graphs of $x(t)$ and $y(t)$ do not cross.



6. **(a)** From Exercise 2, we know that there is only one eigenvalue, $\lambda = 3$, and the eigenvectors (x_0, y_0) satisfy the equation $y_0 = x_0$.
To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\mathbf{V}_1 = \left[\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0$$
$$= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$
$$= \begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}.$$

We obtain the general solution

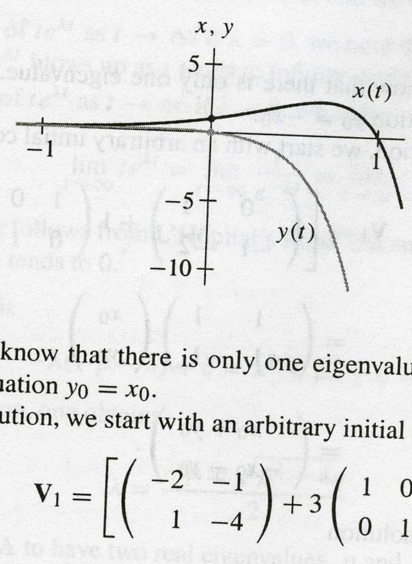
$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{3t}\begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}.$$

(b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Hence, $x(t) = e^{3t}(1 - t)$ and $y(t) = -te^{3t}$.

(c) Compare the graphs of $x(t) = e^{3t}(1-t)$ and $y(t) = -te^{3t}$ with the sketches obtained in part (e) of Exercise 2.



7. **(a)** From Exercise 3, we know that there is only one eigenvalue, $\lambda = -3$, and the eigenvectors (x_0, y_0) satisfy the equation $y_0 = x_0$.
To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\mathbf{V}_1 = \left[\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0$$
$$= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}.$$

We obtain the general solution

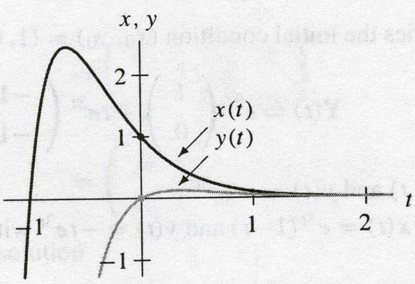
$$\mathbf{Y}(t) = e^{-3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t}\begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}.$$

(b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, $x(t) = e^{-3t}(t+1)$ and $y(t) = te^{-3t}$.

(c) Compare the graphs of $x(t) = e^{-3t}(t+1)$ and $y(t) = te^{-3t}$ with the sketches obtained in part (e) of Exercise 3.



8. **(a)** From Exercise 4, we know that there is only one eigenvalue, $\lambda = -1$, and the eigenvectors (x_0, y_0) satisfy the equation $y_0 = -x_0$.

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\mathbf{V}_1 = \left[\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} + 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} x_0 + y_0 \\ -x_0 - y_0 \end{pmatrix}.$$

We obtain the general solution

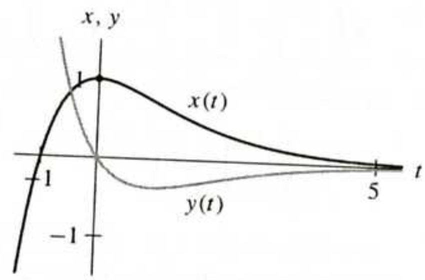
$$\mathbf{Y}(t) = e^{-t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-t}\begin{pmatrix} x_0 + y_0 \\ -x_0 - y_0 \end{pmatrix}.$$

(b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, $x(t) = e^{-t}(t+1)$ and $y(t) = -te^{-t}$.

(c) Compare the graphs of $x(t) = e^{-t}(t+1)$ and $y(t) = -te^{-t}$ with those obtained in part (e) of Exercise 4.



9. **(a)** By solving the quadratic equation, we obtain

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}.$$

Therefore, for the quadratic to have a double root, we must have

$$\alpha^2 - 4\alpha\beta = 0.$$

(b) If zero is a root, we set $\lambda = 0$ in $\lambda^2 + \alpha\lambda + \beta = 0$, and we obtain $\beta = 0$.

10. **(a)** To compute the limit of $te^{\lambda t}$ as $t \to \infty$ if $\lambda > 0$, we note that both t and $e^{\lambda t}$ go to infinity as t goes to infinity. So $te^{\lambda t}$ blows up as t tends to infinity, and the limit does not exist.

(b) To compute the limit of $te^{\lambda t}$ as $t \to \infty$ if $\lambda < 0$, we write

$$\lim_{t\to\infty} te^{\lambda t} = \lim_{t\to\infty} \frac{t}{e^{-\lambda t}} = \lim_{t\to\infty} \frac{1}{-\lambda e^{-\lambda t}}$$

where the last equality follows from L'Hôpital's Rule. Because $e^{-\lambda t}$ tends to infinity as $t \to \infty$ $(-\lambda > 0)$, the fraction tends to 0.

11. The characteristic equation is

$$-\lambda(-p - \lambda) + q = \lambda^2 + p\lambda + q = 0.$$

Solving the quadratic equation, one obtains

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

(a) Therefore, in order for **A** to have two real eigenvalues, p and q must satisfy $p^2 - 4q > 0$.

(b) In order for **A** to have complex eigenvalues, p and q must satisfy $p^2 - 4q < 0$.

(c) In order for **A** to have only one eigenvalue, p and q must satisfy $p^2 - 4q = 0$.

12. The characteristic polynomial of **A** is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - \operatorname{tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

(see Section 3.2). A quadratic polynomial has only one root if and only if its discriminant is 0. In this case, the discriminant of $\det(\mathbf{A} - \lambda\mathbf{I})$ is $\operatorname{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})$.

13. Since every vector is an eigenvector with eigenvalue λ, we substitute $\mathbf{Y} = (1, 0)$ into the equation $\mathbf{AY} = \lambda\mathbf{Y}$ and get

$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence, $a = \lambda$ and $c = 0$. Similarly, letting $\mathbf{Y} = (0, 1)$, we have

$$\begin{pmatrix} b \\ d \end{pmatrix} = \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, $b = 0$ and $d = \lambda$.

14. First note that, because $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are independent, any vector $\mathbf{Y}_3$ can be written as a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$. In other words, there exists k_1 and k_2 such that

$$\mathbf{Y}_3 = k_1\mathbf{Y}_1 + k_1\mathbf{Y}_2.$$

But then

$$\begin{aligned} \mathbf{AY}_3 &= \mathbf{A}(k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2) \\ &= k_1\mathbf{AY}_1 + k_2\mathbf{AY}_2 \\ &= k_1\lambda\mathbf{Y}_1 + k_2\lambda\mathbf{Y}_2 \\ &= \lambda(k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2) \\ &= \lambda\mathbf{Y}_3. \end{aligned}$$

That is, any $\mathbf{Y}_3$ is an eigenvector with eigenvalue λ.

Now use the result of Exercise 13 to conclude that $a = d = \lambda$ and $b = c = 0$.

15. Since $\mathbf{Y}_1(0) = \mathbf{V}_0$ and $\mathbf{Y}_2(0) = \mathbf{W}_0$, we see that $\mathbf{V}_0 = \mathbf{W}_0$.

Evaluating at $t = 1$ yields

$$\mathbf{Y}_1(1) = e^{\lambda}(\mathbf{V}_0 + \mathbf{V}_1) \quad \text{and} \quad \mathbf{Y}_2(1) = e^{\lambda}(\mathbf{W}_0 + \mathbf{W}_1).$$

Since $\mathbf{Y}_1(1) = \mathbf{Y}_2(1)$ and $\mathbf{V}_0 = \mathbf{W}_0$, we see that $\mathbf{V}_1 = \mathbf{W}_1$.

16. **(a)** Since we tend to use λ as the variable in the characteristic polynomial, we will denote the repeated eigenvalue by λ_0. Suppose that

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By assumption, we know that the characteristic polynomial of **A** has λ_0 as a root of multiplicity two. That is,

$$\begin{aligned} \lambda^2 - (a+d)\lambda + (ad - bc) &= (\lambda - \lambda_0)^2 \\ &= \lambda^2 - (2\lambda_0)\lambda + \lambda_0^2. \end{aligned}$$

Therefore, $a + d = 2\lambda_0$, and $ad - bc = \lambda_0^2$.

Now we compute $(\mathbf{A} - \lambda_0\mathbf{I})^2$ using the definition of matrix multiplication. We have

$$(\mathbf{A} - \lambda_0\mathbf{I})^2 = \begin{pmatrix} a - \lambda_0 & b \\ c & d - \lambda_0 \end{pmatrix} \begin{pmatrix} a - \lambda_0 & b \\ c & d - \lambda_0 \end{pmatrix}$$

$$= \begin{pmatrix} (a - \lambda_0)^2 + bc & b(a + d - 2\lambda_0) \\ c(a + d - 2\lambda_0) & bc + (d - \lambda_0)^2 \end{pmatrix}.$$

Since $a + d = 2\lambda_0$, we see that the bottom-left and top-right entries are zero.

Now consider the top-left entry $(a - \lambda_0)^2 + bc$. We have

$$\begin{aligned}(a - \lambda_0)^2 + bc &= a^2 - 2a\lambda_0 + \lambda_0^2 + bc \\ &= a^2 - 2a\lambda_0 + ad - bc + bc,\end{aligned}$$

because $ad - bc = \lambda_0^2$. The right-hand side simplifies to

$$a^2 - 2a\lambda_0 + ad = a(a - 2\lambda_0 + d) = 0$$

because $a + d = 2\lambda_0$.

A similar argument is used to show that the bottom-right entry is zero.

(b) If $\mathbf{V}_0$ is an eigenvector, then $\mathbf{V}_1 = (\mathbf{A} - \lambda_0\mathbf{I})\mathbf{V}_0$ is the zero vector. If not, we use the result of part (a) to compute

$$(\mathbf{A} - \lambda_0\mathbf{I})\mathbf{V}_1 = (\mathbf{A} - \lambda_0\mathbf{I})^2\mathbf{V}_0 = 0 \text{ (the zero vector)}.$$

Consequently, $\mathbf{V}_1$ is an eigenvector.

17. **(a)** The characteristic polynomial is

$$(-\lambda)(-1 - \lambda) + 0 = \lambda^2 + \lambda,$$

so the eigenvalues are $\lambda = 0$ and $\lambda = -1$.

(b) To find the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where $\mathbf{A}$ is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 2y_1 = 0 \\ -y_1 = 0, \end{cases}$$

where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $y_1 = 0$.

To find the eigenvectors $\mathbf{V}_2$ associated to the eigenvalue $\lambda = -1$, we must solve $\mathbf{A}\mathbf{V}_2 = -\mathbf{V}_2$. We get

$$\begin{cases} 2y_2 = -x_2 \\ -y_2 = -y_2. \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = -1$ must satisfy $2y_2 = -x_2$.

he other eigenvalue is nega-
his line as t increases.

responding $x(t)$- and $y(t)$-graphs.

or for each eigenvalue. Using part (b),
eneral solution

$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

), we can substitute $t = 0$ in the general
initial condition is an equilibrium point,

18. **(a)** The characteristic equation is

$$(2-\lambda)(6-\lambda) - 12 = \lambda^2 - 8\lambda = 0.$$

Therefore, the eigenvalues are $\lambda = 0$ and $\lambda = 8$.

(b) To find the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where $\mathbf{A}$ is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 2x_1 + 4y_1 = 0 \\ 3x_1 + 6y_1 = 0, \end{cases}$$

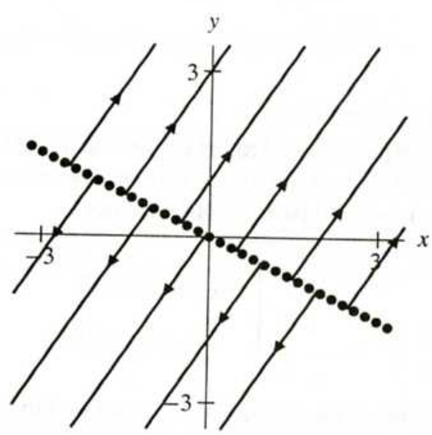
where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $x_1 + 2y_1 = 0$.

To find the eigenvectors $\mathbf{V}_2$ associated to the eigenvalue $\lambda = 8$, we must solve $\mathbf{A}\mathbf{V}_2 = 8\mathbf{V}_2$. We get

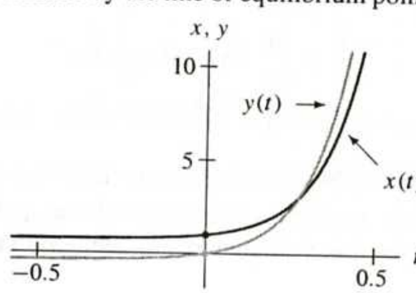
$$\begin{cases} 2x_2 + 4y_2 = 8x_2 \\ 3x_2 + 6y_2 = 8y_2, \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = 8$ must satisfy $2y_2 = 3x_2$.

(c) The equation $x_1 + 2y_1 = 0$ specifies a line of equilibrium points. Since the other eigenvalue is positive, solution curves not corresponding to equilibria move away from this line as t increases.



(d) As t increases, both $x(t)$ and $y(t)$ increase exponentially. As t decreases, both x and y approach constants that are determined by the line of equilibrium points.



(e) To form the general solution, we must pick one eigenvector for each eigenvalue. Using part (b), we pick $\mathbf{V}_1 = (-2, 1)$, and $\mathbf{V}_2 = (2, 3)$. We obtain the general solution

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(f) To determine the solution whose initial condition is $(1, 0)$, we let $t = 0$ in the general solution and obtain the equations

$$k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore, $k_1 = -3/8$ and $k_2 = 1/8$. The particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} \frac{3}{4} + \frac{1}{4}e^{8t} \\ -\frac{3}{8} + \frac{3}{8}e^{8t} \end{pmatrix}.$$

19. **(a)** The characteristic polynomial is

$$(4 - \lambda)(1 - \lambda) - 4 = \lambda^2 - 5\lambda,$$

so the eigenvalues are $\lambda = 0$ and $\lambda = 5$.

(b) To find the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where $\mathbf{A}$ is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 4x_1 + 2y_1 = 0 \\ 2x_1 + y_1 = 0, \end{cases}$$

where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $y_1 = -2x_1$.

To find the eigenvectors $\mathbf{V}_2$ associated to the eigenvalue $\lambda = 5$, we must solve $\mathbf{A}\mathbf{V}_2 = 5\mathbf{V}_2$. We get

$$\begin{cases} 4x_2 + 2y_2 = 5x_2 \\ 2x_2 + y_2 = 5y_2. \end{cases}$$

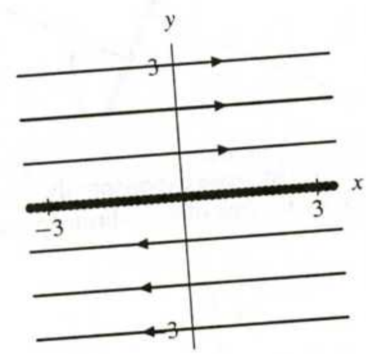
where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = 5$ must satisfy $x_2 = 2y_2$.

(c) The equation $y_1 = -2x_1$ specifies a line of equilibrium points. Since the other eigenvalue is positive, solution curves not corresponding to equilibria move away from this line as t increases.

20. **(a)** The characteristic equation is $\lambda^2 - (a + d)\lambda + ad - bc = 0$. If 0 is an eigenvalue of $\mathbf{A}$, then 0 is a root of the characteristic polynomial. Thus, the constant term in the above equation must be 0—that is, $ad - bc = \det \mathbf{A} = 0$.

(b) If $\det \mathbf{A} = 0$, then the characteristic equation becomes $\lambda^2 - (a+d)\lambda = 0$, and this equation has 0 as a root. Therefore 0 is an eigenvalue of $\mathbf{A}$.

21. **(a)** The characteristic polynomial is $\lambda^2 = 0$, so $\lambda = 0$ is the sole eigenvalue. To sketch the phase portrait we note that $dy/dt = 0$, so $y(t)$ is always a constant function. Moreover, $dx/dt = 2y$, so $x(t)$ is increasing if $y > 0$, and it is decreasing if $y < 0$.



(b) This system is exactly the same as the one in part (a) except that the sign of dx/dt has changed. Hence, the phase portrait is the identical except for the fact that the arrows point the other way.

22. **(a)** This system has only one eigenvalue, $\lambda = 0$, and the eigenvectors lie along the x-axis (the line $y = 0$).

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\mathbf{V}_1 = \left[\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0$$

$$= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}.$$

24. **(a)** The characteristic equation is $\lambda^2+2\lambda+1 = (\lambda+1)^2 = 0$, so the eigenvalue $\lambda = -1$ is repeated. The equilibrium point at the origin is a sink.

(b) To find the associated eigenvectors $\mathbf{V}$, we must solve $\mathbf{AV} = -\mathbf{V}$ where $\mathbf{A}$ is the matrix that defines this linear system. This vector equation is equivalent to the system of scalar equations

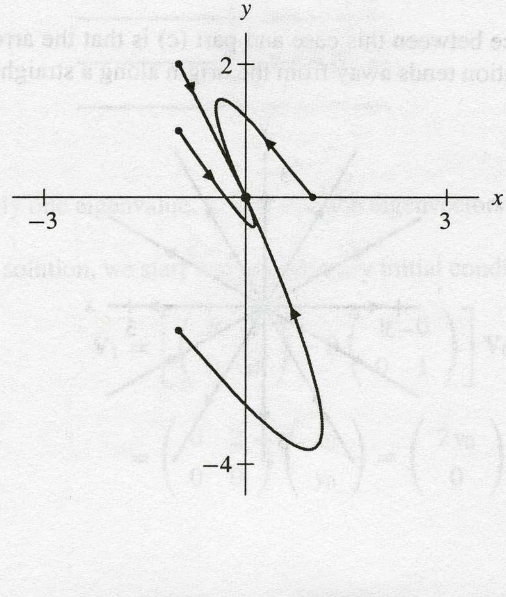
$$\begin{cases} -2x - y = 0 \\ 4x + 2y = 0, \end{cases}$$

so the eigenvectors must satisfy $y = -2x$. One such eigenvector is therefore $(1, -2)$, and all straight-line solutions are of the form

$$\mathbf{Y}(t) = ke^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

where k is an arbitrary constant.

(c) Since this system has only one eigenvalue $\lambda = -1$, we know that the origin is a sink and that all solution curves in the phase plane approach the origin tangent to the line $y = -2x$ of eigenvectors. The direction of approach is determined by the direction field for the system. Solutions with initial conditions that satisfy $y > -2x$ move in a "counter-clockwise" direction and approach the origin in the second quadrant, and solutions with initial conditions that satisfy $y < -2x$ also move in a "counter-clockwise" direction and approach the origin in the fourth quadrant.



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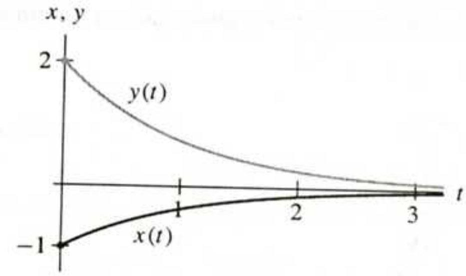
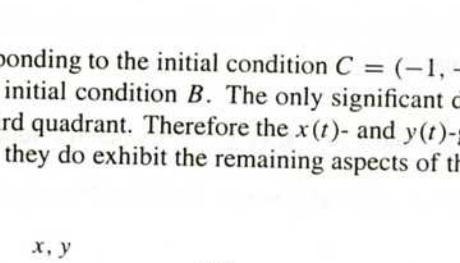
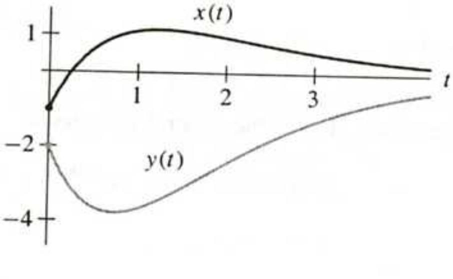
The initial condition $A = (-1, 2)$ is an eigenvector, so the corresponding solution is a straight-line solution. Its $x(t)$- and $y(t)$-graphs are therefore simple exponentials that approach 0 at the rate e^{-t}. We have $y(t) = -2x(t)$ for all t.



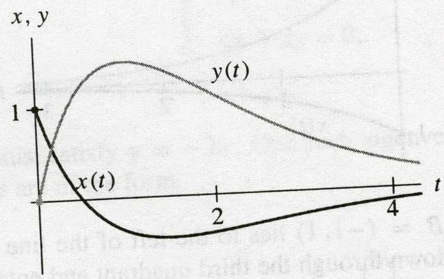
The initial condition $B = (-1, 1)$ lies to the left of the line of eigenvectors. Therefore, its solution curve heads down through the third quadrant and enters the fourth quadrant before it tends to the origin tangent to the line $y = -2x$. The $y(t)$-graph decreases as the $x(t)$-graph increases. We note that $y(t) = 0$ when the solution curve crosses the x-axis, and the two graphs cross when the solution curve crosses the line $y = x$. The function $x(t)$ continues to increase as it becomes positive and attains its maximum value before it tends to 0. The function $y(t)$ assumes a minimum value before it tends to 0.



The solution corresponding to the initial condition $C = (-1, -2)$ behaves in a similar fashion to the solution with initial condition B. The only significant difference is that C is below the line $y = x$ in the third quadrant. Therefore the $x(t)$- and $y(t)$-graphs do not cross as they tend toward 0. However, they do exhibit the remaining aspects of the graphs that correspond to the initial condition B.



The solution corresponding to the initial condition $D = (1, 0)$ moves to the left and up through the first quadrant in the phase plane before it enters the second quadrant and heads toward the origin tangent to the line $y = -2x$. Thus the $y(t)$-graph is always positive for $t > 0$, and it attains a unique maximum value before it tends to 0. Initially the $x(t)$-graph decreases. It crosses the $y(t)$-graph, becomes negative, and attains a minimum value before it tends to 0 as $t \to \infty$.



EXERCISES FOR SECTION 3.6

1. The characteristic polynomial is

$$s^2 + 3s - 10,$$

so the eigenvalues are $s = 2$ and $s = -5$. Hence, the general solution is

$$y(t) = k_1 e^{2t} + k_2 e^{-5t}.$$

2. The characteristic polynomial is

$$s^2 + 6s + 8,$$

so the eigenvalues are $s = -2$ and $s = -4$. Hence, the general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-4t}.$$

3. The characteristic polynomial is

$$s^2 + 6s + 9,$$

so $s = -3$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{-3t} + k_2 t e^{-3t}.$$

4. The characteristic polynomial is

$$s^2 - 4s + 4,$$

so $s = 2$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{2t} + k_2 t e^{2t}.$$

5. The characteristic polynomial is

$$s^2 + 8s + 25,$$

so the complex eigenvalues are $s = -4 \pm 3i$. Hence, the general solution is

$$y(t) = k_1 e^{-4t} \cos 3t + k_2 e^{-4t} \sin 3t.$$

6. The characteristic polynomial is

$$s^2 - 4s + 29,$$

so the complex eigenvalues are $s = 2 \pm 5i$. Hence, the general solution is

$$y(t) = k_1 e^{2t} \cos 5t + k_2 e^{2t} \sin 5t.$$

7. The characteristic polynomial is

$$s^2 + 5s + 6,$$

so the eigenvalues are $s = -2$ and $s = -3$. Hence, the general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-3t},$$

and we have

$$y'(t) = -2k_1 e^{-2t} - 3k_2 e^{-3t}.$$

From the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 0 \\ -2k_1 - 3k_2 = 2. \end{cases}$$

Solving for k_1 and k_2 yields $k_1 = 2$ and $k_2 = -2$. Hence, the solution to our initial-value problem is $y(t) = 2e^{-2t} - 2e^{-3t}$.

8. The characteristic polynomial is

$$s^2 + 4s - 5,$$

so the eigenvalues are $s = 1$ and $s = -5$. Hence, the general solution is

$$y(t) = k_1 e^{t} + k_2 e^{-5t},$$

and we have

$$y'(t) = k_1 e^{t} - 5k_2 e^{-5t}.$$

From the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 11 \\ k_1 - 5k_2 = -7. \end{cases}$$

Solving for k_1 and k_2 yields $k_1 = 8$ and $k_2 = 3$. Hence, the solution to our initial-value problem is $y(t) = 8e^t + 3e^{-5t}$.

9. The characteristic polynomial is

$$s^2 + 2s + 5,$$

so the eigenvalues are $s = -1 \pm 2i$. Hence, the general solution is

$$y(t) = k_1 e^{-t} \cos 3t + k_2 e^{-t} \sin 3t.$$

From the initial condition $y(0) = 3$, we see that $k_1 = 3$. Differentiating

$$y(t) = 3e^{-t} \cos 3t + k_2 e^{-t} \sin 3t$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -3 + 2k_2$. Since $y'(0) = -1$, we have $k_2 = 1$. Hence, the solution to our initial-value problem is

$$y(t) = 3e^{-t} \cos 2t + e^{-t} \sin 2t.$$

10. The characteristic polynomial is

$$s^2 + 4s + 20,$$

so the eigenvalues are $s = -2 \pm 4i$. Hence, the general solution is

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t.$$

From the initial condition $y(0) = 2$, we see that $k_1 = 2$. Differentiating

$$y(t) = 2e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -4 + 4k_2$. Since $y'(0) = -8$, we have $k_2 = -1$. Hence, the solution to our initial-value problem is

$$y(t) = 2e^{-2t} \cos 4t - e^{-2t} \sin 4t.$$

11. The characteristic polynomial is

$$s^2 + 2s + 1,$$

so $s = -1$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{-t} + k_2 t e^{-t}.$$

From the initial condition $y(0) = 1$, we see that $k_1 = 1$. Differentiating

$$y(t) = e^{-t} + k_2 t e^{-t}$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -1 + k_2$. Since $y'(0) = 1$, we have $k_2 = 2$. Hence, the solution to our initial-value problem is

$$y(t) = e^{-t} + 2te^{-t}.$$

12. The characteristic polynomial is

$$s^2 - 4s + 4,$$

so $s = 2$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{2t} + k_2 t e^{2t}.$$

From the initial condition $y(0) = 1$, we see that $k_1 = 1$. Differentiating $y(t) = e^{2t} + k_2 t e^{2t}$ and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = 2 + k_2$. Since $y'(0) = 1$, we have $k_2 = -1$. Hence, the solution to our initial-value problem is

$$y(t) = e^{2t} - te^{2t}.$$

13. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 7y = 0,$$

and the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -7y - 8v.$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 8\lambda + 7 = 0.$$

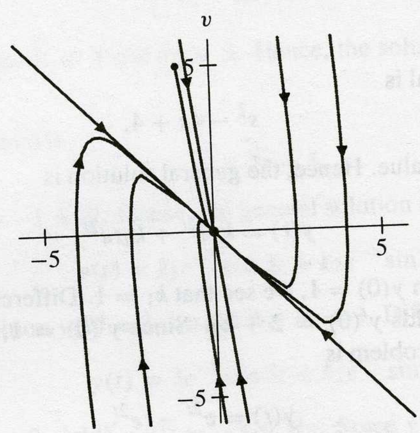
Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -7$.
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = -y \\ -7y - 8v = -v. \end{cases}$$

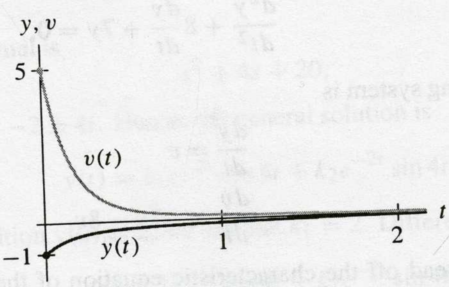
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -7$ must satisfy the equation $v = -7y$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

(d) We know that all solution curves approach the origin as $t \to \infty$ and, with the exception of those whose initial conditions lie on the line $v = -7y$, these solution curves approach the origin tangent to the line $v = -y$.



(e) From the phase portrait, we see that $y(t)$ increases monotonically toward 0 as $t \to \infty$. Also, $v(t)$ decreases monotonically toward 0. It is useful to remember that $v = dy/dt$.



14. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 0,$$

and the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -8y - 6v.$$

(b) Recall that we can read off the characteristic equation of the second-order equation strai from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 6\lambda + 8 = 0.$$

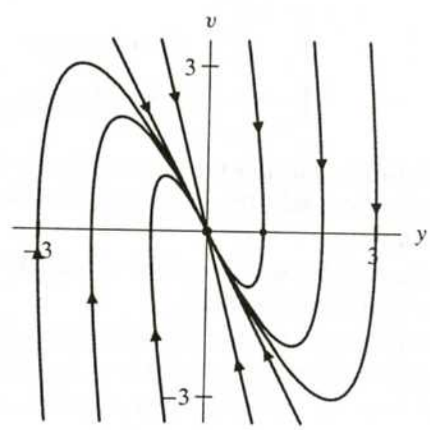
Therefore, the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -2$.
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = -4y \\ -8y - 6v = -4v. \end{cases}$$

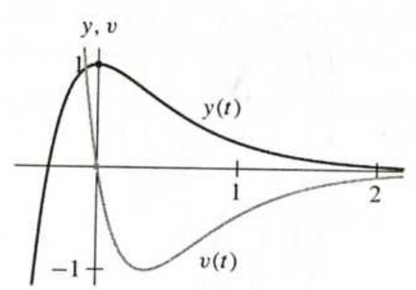
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -4y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -2$ must satisfy the equation $v = -2y$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

(d) We know that all solution curves approach the origin as $t \to \infty$ and, with the exception of those whose initial conditions lie on the line $v = -4y$, these solution curves approach the origin tangent to the line $v = -2y$.



(e) From the phase portrait, we see that $v(t)$ initially decreases from 0 and then increases and tends toward 0 as $t \to \infty$. Also, $y(t)$ decreases monotonically toward 0. It is useful to remember that $v = dy/dt$.



15. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0,$$