3 Determinants

3.1 SOLUTIONS

Notes: If time is needed for other topics, this chapter may be omitted. Section 5.2 contains enough information about determinants to support the discussion there of the characteristic polynomial of a matrix. In section 5.1, some exercises in this section provide practice in computing determinants, while others allow the student to discover the properties of determinants which will be studied in the next section. Determinants are developed through the cofactor expansion, which is given in Theorem 1. Exercises 33–36 in this section provide the first step in the inductive proof of Theorem 3 in the next section.

A "*Checkpoint*" in the *Study Guide* leads students to discover that if the kth column of the identity matrix is replaced by a vector **x**, then the determinant of the resulting matrix is the kth entry of **x**. This idea is used in the proof of Cramer's Rule, in Section 3.3.

1. Expand across along the first row:

$$\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = 3\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 3(-13) + 4(10) = 1$$

Expand down the second column:

$$\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = (-1)^{1+2}\cdot 0\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + (-1)^{2+2}\cdot 3\begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix} + (-1)^{3+2}\cdot 5\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 3(-3) - 5(-2) = 1$$

2. Expand across the first row:

$$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 0\begin{vmatrix} -3 & 0 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix} = -5(4) + 1(22) = 2$$

Expand down the second column:

$$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = (-1)^{1+2}\cdot 5\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} + (-1)^{2+2}\cdot(-3)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + (-1)^{3+2}\cdot 4\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = -5(4) - 3(-2) - 4(-4) = 2$$

3. Expand across the first row:

$$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} - (-4)\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 2(-9) + 4(-5) + (3)(11) = -5$$

10. First expand across the second row, then expand either across the third row or down the second column of the remaining matrix.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-1)^{2+3} \cdot 3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

$$= (-3)\left((-1)^{3+1} \cdot 5 \begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} + (-1)^{3+3} \cdot 4 \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix} \right) = (-3)(5(2)+4(-2)) = -6$$

or

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = (-1)^{2+3} \cdot 3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$$

$$= (-3)\left((-1)^{1+2} \cdot (-2) \begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} + (-1)^{2+2} \cdot (-6) \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} \right) = (-3)(2(-17) - 6(-6)) = -6$$

11. Following the text's instruction, a good strategy is to expand down the first column of the matrix, and repeat the process until the determinant is expressed as the product of the diagonal entries of the original matrix:

$$\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (-1)^{1+1} \cdot 3 \begin{vmatrix} -2 & 3 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{vmatrix} = 3 \cdot (-1)^{1+1} \cdot (-2) \begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = 3(-2)(2) = -12$$

Of course, with Theorem 2 available, the best strategy is to use it and simply compute the product of the diagonal entries in the matrix.

12. Following the text's instruction, a good strategy is to expand along the first row of the matrix, and repeat the process until the determinant is expressed as the product of the diagonal entries of the original matrix:

$$\begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix} = (-1)^{1+1} \cdot 4 \begin{vmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} = 4 \cdot (-1)^{1+1} \cdot (-1) \begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix} = 4(-1)(-9) = 36$$

Of course, with Theorem 2 available, the best strategy is to use it and simply compute the product of the diagonal entries in the matrix.

13. First expand either across the second row or down the second column. Using the second row,

$$\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix} = (-1)^{2+3} \cdot 2 \begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

Now expand down the second column to find:

$$(-1)^{2+3}\cdot 2\begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = -2\left((-1)^{2+2}\cdot 3\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}\right)$$

Now expand either down the first column or across third row. Using the first column,

$$-2\left((-1)^{2+2}\cdot 3\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix}\right) = -6\left((-1)^{1+1}\cdot 4\begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + (-1)^{2+1}\cdot 5\begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix}\right) = (-6)(4(1)-5(1)) = 6$$

14. First expand either across the fourth row or down the fifth column. Using the fifth column,

$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = (-1)^{3+5}\cdot 1\begin{vmatrix} 6 & 3 & 2 & 4 \\ 9 & 0 & -4 & 1 \\ 3 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 \end{vmatrix}$$

Now expand across the third row to find:

$$(-1)^{3+5}\cdot 1\begin{vmatrix} 6 & 3 & 2 & 4 \\ 9 & 0 & -4 & 1 \\ 3 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 \end{vmatrix} = 1\left((-1)^{3+1}\cdot 3\begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}\right)$$

Finally, expand either down the first column or along second row. Using the first column,

$$1\left((-1)^{3+1}\cdot 3\begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix}\right) = 3\left((-1)^{1+1}\cdot 3\begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} + (-1)^{3+1}\cdot 2\begin{vmatrix} 2 & 4 \\ -4 & 1 \end{vmatrix}\right) = (3)(3(-11)+2(18)) = 9$$

15. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = (3)(3)(-1)+(0)(2)(0)+(4)(2)(5)-(0)(3)(4)-(5)(2)(3)-(-1)(2)(0) =$

$-9+0+40-0-30-0=1$

16. $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = (0)(-3)(1)+(5)(0)(2)+(1)(4)(4)-(2)(-3)(1)-(4)(0)(0)-(1)(4)(5) =$

$0+0+16-(-6)-0-20=2$

17. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = (2)(1)(-1)+(-4)(2)(1)+(3)(3)(4)-(1)(1)(3)-(4)(2)(2)-(-1)(3)(-4) =$

$-2+(-8)+36-3-16-12=-5$

35. $E=\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$, $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, $EA=\begin{bmatrix}a+kc & b+kd\\c & d\end{bmatrix}$

det $E = 1$, det $A = ad - bc$,

det $EA = (a+kc)d - c(b+kd) = ad + kcd - bc - kcd = 1(ad - bc) = (\det E)(\det A)$

36. $E=\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$, $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, $EA=\begin{bmatrix}a & b\\ka+c & kb+d\end{bmatrix}$

det $E = 1$, det $A = ad - bc$,

det $EA = a(kb+d) - (ka+c)b = kab + ad - kab - bc = 1(ad - bc) = (\det E)(\det A)$

37. $A=\begin{bmatrix}3 & 1\\4 & 2\end{bmatrix}$, $5A=\begin{bmatrix}15 & 5\\20 & 10\end{bmatrix}$, det $A = 2$, det $5A = 50 \neq 5\det A$

38. $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, $kA=\begin{bmatrix}ka & kb\\kc & kd\end{bmatrix}$, det $A = ad - bc$,

$\det kA = (ka)(kd) - (kb)(kc) = k^2(ad - bc) = k^2 \det A$

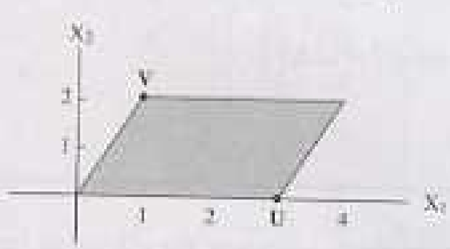
39. a. True. See the paragraph preceding the definition of the determinant.

b. False. See the definition of cofactor, which precedes Theorem 1.

40. a. False. See Theorem 1.

b. False. See Theorem 2.

41. The area of the parallelogram determined by $\mathbf{u}=\begin{bmatrix}3\\0\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{0}$ is 6, since the base of the parallelogram has length 3 and the height of the parallelogram is 2. By the same reasoning, the area of the parallelogram determined by $\mathbf{u}=\begin{bmatrix}3\\0\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x\\2\end{bmatrix}$, $\mathbf{u}+\mathbf{x}$, and $\mathbf{0}$ is also 6.



Also note that $\det[\mathbf{u} \quad \mathbf{v}] = \det\begin{bmatrix}3 & 1\\0 & 2\end{bmatrix} = 6$, and $\det[\mathbf{u} \quad \mathbf{x}] = \det\begin{bmatrix}3 & x\\0 & 2\end{bmatrix} = 6$. The determinant of the matrix whose columns are those vectors which define the sides of the parallelogram adjacent to $\mathbf{0}$ is equal to the area of the parallelogram.

5. $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} = 3$

6. $\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{vmatrix} = 6\begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 3 & -1 \end{vmatrix} = 6\begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (6)(-3) = -18$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 30 & 27 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$

8. $\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$

9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -(-3) = 3$

10. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -(-24) = 24$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5(7) = 35$

16. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3(7) = 21$

17. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -7$

18. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\left(-\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right) = -(-7) = 7$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(7) = 14$

20. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

21. Since $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = -1 \neq 0$, the matrix is invertible.

22. Since $\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 0$, the matrix is not invertible.

23. Since $\begin{vmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix} = 0$, the matrix is not invertible.

24. Since $\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix} = 11 \neq 0$, the columns of the matrix form a linearly independent set.

25. Since $\begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{vmatrix} = -1 \neq 0$, the columns of the matrix form a linearly independent set.

26. Since $\begin{vmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{vmatrix} = 0$, the columns of the matrix form a linearly dependent set.

27. a. True. See Theorem 3.
 b. True. See the paragraph following Example 2.
 c. True. See the paragraph following Theorem 4.
 d. False. See the warning following Example 5.

28. a. True. See Theorem 3.
 b. False. See the paragraphs following Example 2.
 c. False. See Example 3.
 d. False. See Theorem 5.

29. By Theorem 6, $\det B^5 = (\det B)^5 = (-2)^5 = -32$.

30. Suppose the two rows of a square matrix A are equal. By swapping these two rows, the matrix A is not changed so its determinant should not change. But since swapping rows changes the sign of the determinant, $\det A = -\det A$. This is only possible if $\det A = 0$. The same may be proven true for columns by applying the above result to A^T and using Theorem 5.

31. By Theorem 6, $(\det A)(\det A^{-1}) = \det I = 1$, so $\det A^{-1} = 1/\det A$.

32. By factoring an r out of each of the n rows, $\det(rA) = r^n \det A$.

33. By Theorem 6, $\det AB = (\det A)(\det B) = (\det B)(\det A) = \det BA$.

34. By Theorem 6 and Exercise 31,

$$\begin{aligned}\det(PAP^{-1}) &= (\det P)(\det A)(\det P^{-1}) = (\det P)(\det P^{-1})(\det A) \\ &= (\det P)\left(\frac{1}{\det P}\right)(\det A) = 1\det A \\ &= \det A\end{aligned}$$

35. By Theorem 6 and Theorem 5, $\det U^TU = (\det U^T)(\det U) = (\det U)^2$. Since $U^TU = I$, $\det U^TU = \det I = 1$, so $(\det U)^2 = 1$. Thus $\det U = \pm 1$.

36. By Theorem 6 $\det A^4 = (\det A)^4$. Since $\det A^4 = 0$, then $(\det A)^4 = 0$. Thus $\det A = 0$, and A is not invertible by Theorem 4.

37. By Theorem 2, $\det A = 3$ and $\det B = 8$, while $AB = \begin{bmatrix} 6 & 0 \\ 17 & 4 \end{bmatrix}$. Thus $\det AB = 24 = 3 \times 8 = (\det A)(\det B)$.

38. Compute $\det A = 0$ and $\det B = -2$. Also, $AB = \begin{bmatrix} 6 & 0 \\ -2 & 0 \end{bmatrix}$. Thus $\det AB = 0 =$ $(\det A)(\det B)$.

39. a. By Theorem 6, $\det AB = (\det A)(\det B) = 4\times -3 = -12$.

b. By Exercise 32, $\det 5A = 5^3\det A = 125\times 4 = 500$.

c. By Theorem 5, $\det B^T = \det B = -3$.

d. By Exercise 31, $\det A^{-1} = 1/\det A = 1/4$.

e. By Theorem 6, $\det A^3 = (\det A)^3 = 4^3 = 64$.

40. a. By Theorem 6, $\det AB = (\det A)(\det B) = -1\times 2 = -2$.

b. By Theorem 6, $\det B^5 = (\det B)^5 = 2^5 = 32$.

c. By Exercise 32, $\det 2A = 2^4\det A = 16\times -1 = -16$.

d. By Theorems 5 and 6, $\det A^TA = (\det A^T)(\det A) = (\det A)(\det A) = -1\times -1 = 1$.

e. By Theorem 6 and Exercise 31,
$\det B^{-1}AB = (\det B^{-1})(\det A)(\det B) = (1/\det B)(\det A)(\det B) = \det A = -1$.

41. $\det A = (a+e)d - c(b+f) = ad + ed - bc - cf = (ad - bc) + (ed - cf) = \det B + \det C$.

42. $\det(A+B) = \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = (1+a)(1+d) - cb = 1 + a + d + ad - cb = \det A + a + d + \det B$, so $\det(A+B) = \det A + \det B$ if and only if $a + d = 0$.

43. Compute $\det A$ by using a cofactor expansion down the third column:

$$\begin{aligned}\det A &= (u_1+v_1)\det A_{13} - (u_2+v_2)\det A_{23} + (u_3+v_3)\det A_{33} \\ &= u_1\det A_{13} - u_2\det A_{23} + u_3\det A_{33} + v_1\det A_{13} - v_2\det A_{23} + v_3\det A_{33} \\ &= \det B + \det C\end{aligned}$$

44. By Theorem 5, $\det AE = \det(AE)^T$. Since $(AE)^T = E^TA^T$, $\det AE = \det(E^TA^T)$. Now E^T is itself an elementary matrix, so by the proof of Theorem 3, $\det(E^TA^T) = (\det E^T)(\det A^T)$. Thus it is true that $\det AE = (\det E^T)(\det A^T)$, and by applying Theorem 5, $\det AE = (\det E)(\det A)$.

45. [M] Answers will vary, but will show that $\det A^TA$ always equals 0 while $\det AA^T$ should seldom be zero. To see why A^TA should not be invertible (and thus $\det A^TA = 0$), let A be a matrix with more columns than rows. Then the columns of A must be linearly dependent, so the equation $A\mathbf{x} = \mathbf{0}$ must have a non-trivial solution $\mathbf{x}$. Thus $(A^TA)\mathbf{x} = A^T(A\mathbf{x}) = A^T\mathbf{0} = \mathbf{0}$, and the equation $(A^TA)\mathbf{x} = \mathbf{0}$ has a non-trivial solution. Since A^TA is a square matrix, the Invertible Matrix Theorem now says that A^TA is not invertible. Notice that the same argument will not work in general for AA^T, since A^T has more rows than columns, so its columns are not automatically linearly dependent.

46. [M] Compute $\det A = 1$ and $\operatorname{cond} A = 23683$. Note that this is the ℓ_2 condition number, which is used in Section 2.3. Since $\det A \neq 0$, it is invertible and

$$A^{-1} = \begin{bmatrix} -19 & -14 & 0 & 7 \\ -549 & -401 & -2 & 196 \\ 267 & 195 & 1 & -95 \\ -278 & -203 & -1 & 99 \end{bmatrix}$$

Since $\det A = 15s^2 + 45 = 15(s^2+3) \neq 0$ for all values of s, the system will have a unique solution for all values of s. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{15s+10}{15(s^2+3)} = \frac{3s+2}{3(s^2+3)},\ x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{6s-27}{15(s^2+3)} = \frac{2s-9}{5(s^2+3)}.$$

9. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} s & -2s \\ 3 & 6s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Compute

$$A_1(\mathbf{b}) = \begin{bmatrix} -1 & -2s \\ 4 & 6s \end{bmatrix},\ A_2(\mathbf{b}) = \begin{bmatrix} s & -1 \\ 3 & 4 \end{bmatrix},\ \det A_1(\mathbf{b}) = 2s,\ \det A_2(\mathbf{b}) = 4s+3.$$

Since $\det A = 6s^2 + 6s = 6s(s+1) = 0$ for $s = 0, -1$, the system will have a unique solution when $s \neq 0, -1$. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{2s}{6s(s+1)} = \frac{1}{3(s+1)},\ x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{4s+3}{6s(s+1)}.$$

10. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2s & 1 \\ 3s & 6s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 1 & 1 \\ 2 & 6s \end{bmatrix},\ A_2(\mathbf{b}) = \begin{bmatrix} 2s & 1 \\ 3s & 2 \end{bmatrix},\ \det A_1(\mathbf{b}) = 6s-2,\ \det A_2(\mathbf{b}) = s.$$

Since $\det A = 12s^2 - 3s = 3s(4s-1) = 0$ for $s = 0, 1/4$, the system will have a unique solution when $s \neq 0, 1/4$. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{6s-2}{3s(4s-1)},\ x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{s}{3s(4s-1)} = \frac{1}{3(4s-1)}.$$

11. Since $\det A = 3$ and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0,\quad C_{12} = -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = -3,\quad C_{13} = \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{21} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = 1,\quad C_{22} = \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1,\quad C_{23} = -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} = 0,\quad C_{32} = -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3,\quad C_{33} = \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} = 6,$$

$$\operatorname{adj} A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{bmatrix}.$$

12. Since $\det A = 5$ and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = -1,\quad C_{12} = -\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 0,\quad C_{13} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 2,$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 3,\quad C_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0,\quad C_{23} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1,$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7,\quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5,\quad C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -4.$$

Hence by Theorem 9, $P = [\text{Re}\,\mathbf{v}_1 \quad \text{Im}\,\mathbf{v}_1 \quad \text{Re}\,\mathbf{v}_2 \quad \text{Im}\,\mathbf{v}_2] = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}$ and

$C = \begin{bmatrix} -.4 & -1 & 0 & 0 \\ 1 & -.4 & 0 & 0 \\ 0 & 0 & -.2 & -.5 \\ 0 & 0 & .5 & -.2 \end{bmatrix}$. Other choices are possible, but C must equal $P^{-1}AP$.

5.6 SOLUTIONS

1. The exercise does not specify the matrix A, but only lists the eigenvalues 3 and 1/3, and the corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Also, $\mathbf{x}_0 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

   a. To find the action of A on $\mathbf{x}_0$, express $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, find c_1 and c_2 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. This is certainly possible because the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (by inspection and also because they correspond to distinct eigenvalues) and hence form a basis for $\mathbf{R}^2$. (Two linearly independent vectors in $\mathbf{R}^2$ automatically span $\mathbf{R}^2$.) The row reduction $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}_0] = \begin{bmatrix} 1 & -1 & 9 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \end{bmatrix}$ shows that $\mathbf{x}_0 = 5\mathbf{v}_1 - 4\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors (for the eigenvalues 3 and 1/3):

   $$\mathbf{x}_1 = A\mathbf{x}_0 = 5A\mathbf{v}_1 - 4A\mathbf{v}_2 = 5\cdot 3\mathbf{v}_1 - 4\cdot(1/3)\mathbf{v}_2 = \begin{bmatrix} 15 \\ 15 \end{bmatrix} - \begin{bmatrix} -4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 49/3 \\ 41/3 \end{bmatrix}$$

   b. Each time A acts on a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, the $\mathbf{v}_1$ term is multiplied by the eigenvalue 3 and the $\mathbf{v}_2$ term is multiplied by the eigenvalue 1/3:

   $$\mathbf{x}_2 = A\mathbf{x}_1 = A[5\cdot 3\mathbf{v}_1 - 4(1/3)\mathbf{v}_2] = 5(3)^2\mathbf{v}_1 - 4(1/3)^2\mathbf{v}_2$$

   In general, $\mathbf{x}_k = 5(3)^k\mathbf{v}_1 - 4(1/3)^k\mathbf{v}_2$, for $k \geq 0$.

2. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$ are eigenvectors of a 3×3 matrix A, corresponding to eigenvalues 3, 4/5, and 3/5, respectively. Also, $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. To describe the solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 1, 2, \ldots)$, first write $\mathbf{x}_0$ in terms of the eigenvectors.

   $$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{x}_0] = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ -3 & -5 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow \mathbf{x}_0 = 2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$$

7. Since $\mathbf{w}=\begin{bmatrix}3\\-1\\-5\end{bmatrix}$, $\|\mathbf{w}\|=\sqrt{\mathbf{w}\cdot\mathbf{w}}=\sqrt{3^2+(-1)^2+(-5)^2}=\sqrt{35}$.

8. Since $\mathbf{x}=\begin{bmatrix}6\\-2\\3\end{bmatrix}$, $\|\mathbf{x}\|=\sqrt{\mathbf{x}\cdot\mathbf{x}}=\sqrt{6^2+(-2)^2+3^2}=\sqrt{49}=7$.

9. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(-30)^2+40^2}}\begin{bmatrix}-30\\40\end{bmatrix}=\frac{1}{50}\begin{bmatrix}-30\\40\end{bmatrix}=\begin{bmatrix}-3/5\\4/5\end{bmatrix}$$

10. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(-6)^2+4^2+(-3)^2}}\begin{bmatrix}-6\\4\\-3\end{bmatrix}=\frac{1}{\sqrt{61}}\begin{bmatrix}-6\\4\\-3\end{bmatrix}=\begin{bmatrix}-6/\sqrt{61}\\4/\sqrt{61}\\-3\sqrt{61}\end{bmatrix}$$

11. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(7/4)^2+(1/2)^2+1^2}}\begin{bmatrix}7/4\\1/2\\1\end{bmatrix}=\frac{1}{\sqrt{69/16}}\begin{bmatrix}7/4\\1/2\\1\end{bmatrix}=\begin{bmatrix}7/\sqrt{69}\\2/\sqrt{69}\\4/\sqrt{69}\end{bmatrix}$$

12. A unit vector in the direction of the given vector is

$$\frac{1}{\sqrt{(8/3)^2+2^2}}\begin{bmatrix}8/3\\2\end{bmatrix}=\frac{1}{\sqrt{100/9}}\begin{bmatrix}8/3\\2\end{bmatrix}=\begin{bmatrix}4/5\\3/5\end{bmatrix}$$

13. Since $\mathbf{x}=\begin{bmatrix}10\\-3\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}-1\\-5\end{bmatrix}$, $\|\mathbf{x}-\mathbf{y}\|^2=[10-(-1)]^2+[-3-(-5)]^2=125$ and $\text{dist}(\mathbf{x},\mathbf{y})=\sqrt{125}=5\sqrt{5}$.

14. Since $\mathbf{u}=\begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $\mathbf{z}=\begin{bmatrix}-4\\-1\\8\end{bmatrix}$, $\|\mathbf{u}-\mathbf{z}\|^2=[0-(-4)]^2+[-5-(-1)]^2+[2-8]^2=68$ and $\text{dist}(\mathbf{u},\mathbf{z})=\sqrt{68}=2\sqrt{17}$.

15. Since $\mathbf{a}\cdot\mathbf{b}=8(-2)+(-5)(-3)=-1\neq 0$, **a** and **b** are not orthogonal.

16. Since $\mathbf{u}\cdot\mathbf{v}=12(2)+(3)(-3)+(-5)(3)=0$, **u** and **v** are orthogonal.

17. Since $\mathbf{u}\cdot\mathbf{v}=3(-4)+2(1)+(-5)(-2)+0(6)=0$, **u** and **v** are orthogonal.

18. Since $\mathbf{y}\cdot\mathbf{z}=(-3)(1)+7(-8)+4(15)+0(-7)=1\neq 0$, **y** and **z** are not orthogonal.

19. a. True. See the definition of $\|\mathbf{v}\|$.

b. True. See Theorem 1(c).

c. True. See the discussion of Figure 5.

d. False. Counterexample: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

e. True. See the box following Example 6.

20. a. True. See Example 1 and Theorem 1(a).
b. False. The absolute value sign is missing. See the box before Example 2.
c. True. See the defintion of orthogonal complement.
d. True. See the Pythagorean Theorem.
e. True. See Theorem 3.

21. Theorem 1(b):

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}+\mathbf{v})^T\mathbf{w} = (\mathbf{u}^T+\mathbf{v}^T)\mathbf{w} = \mathbf{u}^T\mathbf{w}+\mathbf{v}^T\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$$

The second and third equalities used Theorems 3(b) and 2(c), respectively, from Section 2.1.
Theorem 1(c):

$$(c\mathbf{u})\cdot\mathbf{v} = (c\mathbf{u})^T\mathbf{v} = c(\mathbf{u}^T\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$$

The second and third equalities used Theorems 3(c) and 2(d), respectively, from Section 2.1.

22. Since $\mathbf{u}\cdot\mathbf{u}$ is the sum of the squares of the entries in $\mathbf{u}$, $\mathbf{u}\cdot\mathbf{u} \ge 0$. The sum of squares of numbers is zero if and only if all the numbers are themselves zero.

23. One computes that $\mathbf{u}\cdot\mathbf{v} = 2(-7)+(-5)(-4)+(-1)6 = 0$, $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u} = 2^2+(-5)^2+(-1)^2 = 30$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v} = (-7)^2+(-4)^2+6^2 = 101$, and $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (2+(-7))^2+(-5+(-4))^2+(-1+6)^2 = 131$.

24. One computes that

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2$$

and

$$\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u}-2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2-2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2$$

so

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2+\|\mathbf{u}\|^2-2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2$$

25. When $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, the set H of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to v is the subspace of vectors whose entries satisfy $ax+by=0$. If $a \ne 0$, then $x = -(b/a)y$ with y a free variable, and H is a line through the origin. A natural choice for a basis for H in this case is $\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\}$. If $a = 0$ and $b \ne 0$, then $by = 0$. Since $b \ne 0$, $y = 0$ and x is a free variable. The subspace H is again a line through the origin. A natural choice for a basis for H in this case is $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, but $\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\}$ is still a basis for H since $a = 0$ and $b \ne 0$. If $a = 0$ and $b = 0$, then $H = \mathbb{R}^2$ since the equation $0x + 0y = 0$ places no restrictions on x or y.

26. Theorem 2 in Chapter 4 may be used to show that W is a subspace of $\mathbb{R}^3$, because W is the null space of the 1×3 matrix $\mathbf{u}^T$. Geometrically, W is a plane through the origin.

27. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{y}\cdot\mathbf{u}=\mathbf{y}\cdot\mathbf{v}=0$, and hence by a property of the inner product, $\mathbf{y}\cdot(\mathbf{u}+\mathbf{v})=\mathbf{y}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v}=0+0=0$. Thus $\mathbf{y}$ is orthogonal to $\mathbf{u}+\mathbf{v}$.

28. An arbitrary $\mathbf{w}$ in Span{$\mathbf{u}$, $\mathbf{v}$} has the form $\mathbf{w}=c_1\mathbf{u}+c_2\mathbf{v}$. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{y}=\mathbf{v}\cdot\mathbf{y}=0$. By Theorem 1(b) and 1(c),

$$\mathbf{w}\cdot\mathbf{y}=(c_1\mathbf{u}+c_2\mathbf{v})\cdot\mathbf{y}=c_1(\mathbf{u}\cdot\mathbf{y})+c_2(\mathbf{v}\cdot\mathbf{y})=0+0=0$$

29. A typical vector in W has the form $\mathbf{w}=c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p$. If $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, then by Theorems 1(b) and 1(c),

$$\mathbf{w}\cdot\mathbf{x}=(c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p)\cdot\mathbf{y}=c_1(\mathbf{v}_1\cdot\mathbf{x})+\ldots+c_p(\mathbf{v}_p\cdot\mathbf{x})=0$$

So $\mathbf{x}$ is orthogonal to each $\mathbf{w}$ in W.

30. **a.** If $\mathbf{z}$ is in $W^\perp$, $\mathbf{u}$ is in W, and c is any scalar, then $(c\mathbf{z})\cdot\mathbf{u}=c(\mathbf{z}\cdot\mathbf{u})=c0=0$. Since $\mathbf{u}$ is any element of W, $c\mathbf{z}$ is in $W^\perp$.

b. Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be in $W^\perp$. Then for any $\mathbf{u}$ in W, $(\mathbf{z}_1+\mathbf{z}_2)\cdot\mathbf{u}=\mathbf{z}_1\cdot\mathbf{u}+\mathbf{z}_2\cdot\mathbf{u}=0+0=0$. Thus $\mathbf{z}_1+\mathbf{z}_2$ is in $W^\perp$.

c. Since $\mathbf{0}$ is orthogonal to every vector, $\mathbf{0}$ is in $W^\perp$. Thus $W^\perp$ is a subspace.

31. Suppose that $\mathbf{x}$ is in W and $W^\perp$. Since $\mathbf{x}$ is in $W^\perp$, $\mathbf{x}$ is orthogonal to every vector in W, including $\mathbf{x}$ itself. So $\mathbf{x}\cdot\mathbf{x}=0$, which happens only when $\mathbf{x}=\mathbf{0}$.

32. [M]

a. One computes that $\|\mathbf{a}_1\|=\|\mathbf{a}_2\|=\|\mathbf{a}_3\|=\|\mathbf{a}_4\|=1$ and that $\mathbf{a}_i\cdot\mathbf{a}_j=0$ for $i\neq j$.

b. Answers will vary, but it should be that $\|A\mathbf{u}\|=\|\mathbf{u}\|$ and $\|A\mathbf{v}\|=\|\mathbf{v}\|$.

c. Answers will again vary, but the cosines should be equal.

d. A conjecture is that multiplying by A does not change the lengths of vectors or the angles between vectors.

33. [M] Answers to the calculations will vary, but will demonstrate that the mapping $\mathbf{x}\mapsto T(\mathbf{x})=\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ (for $\mathbf{v}\neq\mathbf{0}$) is a linear transformation. To confirm this, let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$, and let c be any scalar. Then

$$T(\mathbf{x}+\mathbf{y})=\left(\frac{(\mathbf{x}+\mathbf{y})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=\left(\frac{(\mathbf{x}\cdot\mathbf{v})+(\mathbf{y}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}+\left(\frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=T(\mathbf{x})+T(\mathbf{y})$$

and

$$T(c\mathbf{x})=\left(\frac{(c\mathbf{x})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=\left(\frac{c(\mathbf{x}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=c\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=cT(\mathbf{x})$$

34. [M] One finds that

$$N=\begin{bmatrix}-5&1\\-1&4\\1&0\\0&-1\\0&3\end{bmatrix},\ R=\begin{bmatrix}1&0&5&0&-1/3\\0&1&1&0&-4/3\\0&0&0&1&1/3\end{bmatrix}$$

8. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -6 + 6 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent by Theorem 4. Two such vectors in $\mathbb{R}^2$ automatically form a basis for $\mathbb{R}^2$. So $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = -\frac{3}{2}\mathbf{u}_1 + \frac{3}{4}\mathbf{u}_2$$

9. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent by Theorem 4. Three such vectors in $\mathbb{R}^3$ automatically form a basis for $\mathbb{R}^3$. So $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{5}{2}\mathbf{u}_1 - \frac{3}{2}\mathbf{u}_2 + 2\mathbf{u}_3$$

10. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent by Theorem 4. Three such vectors in $\mathbb{R}^3$ automatically form a basis for $\mathbb{R}^3$. So $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3$$

11. Let $\mathbf{y} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. The orthogonal projection of $\mathbf{y}$ onto the line through $\mathbf{u}$ and the origin is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$, and this vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{1}{2}\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

12. Let $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. The orthogonal projection of $\mathbf{y}$ onto the line through $\mathbf{u}$ and the origin is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$, and this vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = -\frac{2}{5}\mathbf{u} = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}$$

13. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = -\frac{13}{65}\mathbf{u} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}$$

The component of $\mathbf{y}$ orthogonal to $\mathbf{u}$ is

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$$

Thus $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}}) = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$.

14. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{2}{5}\mathbf{u} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

The component of **y** orthogonal to **u** is

$$\mathbf{y}-\hat{\mathbf{y}}=\begin{bmatrix}-4/5\\28/5\end{bmatrix}$$

Thus $\mathbf{y}=\hat{\mathbf{y}}+(\mathbf{y}-\hat{\mathbf{y}})=\begin{bmatrix}14/5\\2/5\end{bmatrix}+\begin{bmatrix}-4/5\\28/5\end{bmatrix}$.

15. The distance from **y** to the line through **u** and the origin is $\|\mathbf{y}-\hat{\mathbf{y}}\|$. One computes that

$$\mathbf{y}-\hat{\mathbf{y}}=\mathbf{y}-\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\begin{bmatrix}3\\1\end{bmatrix}-\frac{3}{10}\begin{bmatrix}8\\6\end{bmatrix}=\begin{bmatrix}3/5\\-4/5\end{bmatrix}$$

so $\|\mathbf{y}-\hat{\mathbf{y}}\|=\sqrt{9/25+16/25}=1$ is the desired distance.

16. The distance from **y** to the line through **u** and the origin is $\|\mathbf{y}-\hat{\mathbf{y}}\|$. One computes that

$$\mathbf{y}-\hat{\mathbf{y}}=\mathbf{y}-\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\begin{bmatrix}-3\\9\end{bmatrix}-3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-6\\3\end{bmatrix}$$

so $\|\mathbf{y}-\hat{\mathbf{y}}\|=\sqrt{36+9}=3\sqrt{5}$ is the desired distance.

17. Let $\mathbf{u}=\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}-1/2\\0\\1/2\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. However, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1/3$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1/2$, so $\{\mathbf{u},\mathbf{v}\}$ is not an orthonormal set. The vectors **u** and **v** may be normalized to form the orthonormal set

$$\left\{\frac{\mathbf{u}}{\|\mathbf{u}\|},\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\}=\left\{\begin{bmatrix}\sqrt{3}/3\\\sqrt{3}/3\\\sqrt{3}/3\end{bmatrix},\begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}\right\}$$

18. Let $\mathbf{u}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=-1\neq 0$, $\{\mathbf{u},\mathbf{v}\}$ is not an orthogonal set.

19. Let $\mathbf{u}=\begin{bmatrix}-.6\\.8\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}.8\\.6\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, so $\{\mathbf{u},\mathbf{v}\}$ is an orthonormal set.

20. Let $\mathbf{u}=\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}1/3\\2/3\\0\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\{\mathbf{u},\mathbf{v}\}$ is an orthogonal set. However, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$ and $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=5/9$, so $\{\mathbf{u},\mathbf{v}\}$ is not an orthonormal set. The vectors **u** and **v** may be normalized to form the orthonormal set

$$\left\{\frac{\mathbf{u}}{\|\mathbf{u}\|},\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\}=\left\{\begin{bmatrix}-2/3\\1/3\\2/3\end{bmatrix},\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}\right\}$$

21. Let $\mathbf{u}=\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{20}\\3/\sqrt{20}\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{20}\\-1/\sqrt{20}\end{bmatrix}$, and $\mathbf{w}=\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{w}=0$, $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$, $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, and $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=1$, so $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an orthonormal set.

22. Let $\mathbf{u}=\begin{bmatrix}1/\sqrt{18}\\4/\sqrt{18}\\1/\sqrt{18}\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix}$, and $\mathbf{w}=\begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix}$. Since $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}=\mathbf{v}\cdot\mathbf{w}=0$, $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an orthogonal set. Also, $\|\mathbf{u}\|^2=\mathbf{u}\cdot\mathbf{u}=1$, $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}=1$, and $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=1$, so $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is an orthonormal set.

23. a. True. For example, the vectors $\mathbf{u}$ and $\mathbf{y}$ in Example 3 are linearly independent but not orthogonal.
b. True. The formulas for the weights are given in Theorem 5.
c. False. See the paragraph following Example 5.
d. False. The matrix must also be square. See the paragraph before Example 7.
e. False. See Example 4. The distance is $\|\mathbf{y}-\hat{\mathbf{y}}\|$.

24. a. True. But every orthogonal set of *nonzero vectors* is linearly independent. See Theorem 4.
b. False. To be orthonormal, the vectors is S must be unit vectors as well as being orthogonal to ea
other.
c. True. See Theorem 7(a).
d. True. See the paragraph before Example 3.
e. True. See the paragraph before Example 7.

25. To prove part (b), note that

$$(U\mathbf{x})\cdot(U\mathbf{y})=(U\mathbf{x})^T(U\mathbf{y})=\mathbf{x}^TU^TU\mathbf{y}=\mathbf{x}^T\mathbf{y}=\mathbf{x}\cdot\mathbf{y}$$

because $U^TU=I$. If $\mathbf{y}=\mathbf{x}$ in part (b), $(U\mathbf{x})\cdot(U\mathbf{x})=\mathbf{x}\cdot\mathbf{x}$, which implies part (a). Part (c) of the Th
follows immediately fom part (b).

26. A set of n nonzero orthogonal vectors must be linearly independent by Theorem 4, so if such a set
W it is a basis for W. Thus W is an n-dimensional subspace of $\mathbb{R}^n$, and $W=\mathbb{R}^n$.

27. If U has orthonormal columns, then $U^TU=I$ by Theorem 6. If U is also a square matrix, then th
equation $U^TU=I$ implies that U is invertible by the Invertible Matrix Theorem.

28. If U is an $n\times n$ orthogonal matrix, then $I=UU^{-1}=UU^T$. Since U is the transpose of U^T, Theor
applied to U^T says that U^T has orthogonal columns. In particular, the columns of U^T are linear
independent and hence form a basis for $\mathbb{R}^n$ by the Invertible Matrix Theorem. That is, the rows o
a basis (an orthonormal basis) for $\mathbb{R}^n$.

29. Since U and V are orthogonal, each is invertible. By Theorem 6 in Section 2.2, UV is invertible a
$(UV)^{-1}=V^{-1}U^{-1}=V^TU^T=(UV)^T$, where the final equality holds by Theorem 3 in Section 2.1.
is an orthogonal matrix.