

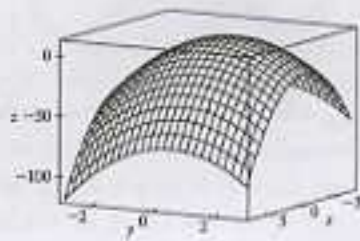
To verify our predictions, we have  $f(x, y) = 4 + x^2 + y^2 - 3xy \Rightarrow f_x(x, y) = 3x^2 - 3y$ ,  $f_y(x, y) = 3y^2 - 3x$ . We have critical points where these partial derivatives are equal to 0:  $3x^2 - 3y = 0$ ,  $3y^2 - 3x = 0$ . Substituting  $y = x^2$  from the first equation into the second equation gives  $3(x^2)^2 - 3x = 0 \Rightarrow 3x(x^3 - 1) = 0 \Rightarrow x = 0$  or  $x = 1$ . Then we have two critical points,  $(0, 0)$  and  $(1, 1)$ . The second partial derivatives are  $f_{xx}(x, y) = 6x$ ,  $f_{xy}(x, y) = -3$ , and  $f_{yy}(x, y) = 6y$ , so  $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$ . Then  $D(0, 0) = 36(0)(0) - 9 = -9$ , and  $D(1, 1) = 36(1)(1) - 9 = 27$ . Since  $D(0, 0) < 0$ ,  $f$  has a saddle point at  $(0, 0)$  by the Second Derivatives Test. Since  $D(1, 1) > 0$  and  $f_{xx}(1, 1) > 0$ ,  $f$  has a local minimum at  $(1, 1)$ .

4. In the figure, points at approximately  $(-1, 1)$  and  $(-1, -1)$  are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of  $f$  are increasing. Hence we would expect local minima at or near  $(-1, \pm 1)$ . Similarly, the point  $(1, 0)$  appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of  $f$  are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points  $(-1, 0)$ ,  $(1, 1)$ , and  $(1, -1)$ . The values of  $f$  increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

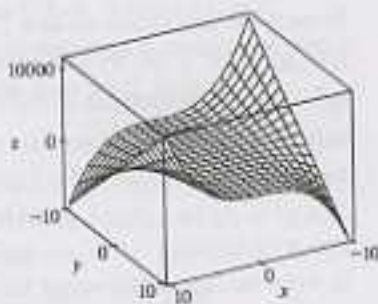
To confirm our predictions, we have  $f(x, y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x, y) = 3 - 3x^2$ ,  $f_y(x, y) = -4y + 4y^3$ . Setting these partial derivatives equal to 0, we have  $3 - 3x^2 = 0 \Rightarrow x = \pm 1$  and  $-4y + 4y^3 = 0 \Rightarrow y(y^2 - 1) = 0 \Rightarrow y = 0, \pm 1$ . So our critical points are  $(\pm 1, 0)$ ,  $(\pm 1, \pm 1)$ . The second partial derivatives are  $f_{xx}(x, y) = -6x$ ,  $f_{xy}(x, y) = 0$ , and  $f_{yy}(x, y) = 12y^2 - 4$ , so  $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x$ . We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	$D$	$f_{xx}$	Conclusion
$(1, 0)$	24	-6	$D > 0, f_{xx} < 0 \Rightarrow f$ has a local maximum at $(1, 0)$
$(1, 1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1, 1)$
$(1, -1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1, -1)$
$(-1, 0)$	-24		$D < 0 \Rightarrow f$ has a saddle point at $(-1, 0)$
$(-1, 1)$	48	6	$D > 0, f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1, 1)$
$(-1, -1)$	48	6	$D > 0, f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1, -1)$

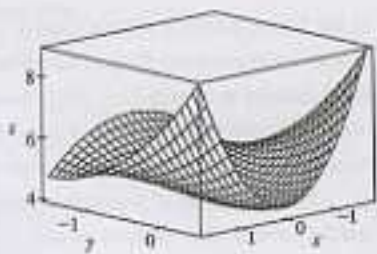
5.  $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2 \Rightarrow f_x = -2 - 2x$ ,  $f_y = 4 - 8y$ ,  $f_{xx} = -2$ ,  $f_{xy} = 0$ ,  $f_{yy} = -8$ . Then  $f_x = 0$  and  $f_y = 0$  imply  $x = -1$  and  $y = \frac{1}{2}$ , and the only critical point is  $(-1, \frac{1}{2})$ .  $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$ , and since  $D(-1, \frac{1}{2}) = 16 > 0$  and  $f_{xx}(-1, \frac{1}{2}) = -2 < 0$ ,  $f(-1, \frac{1}{2}) = 11$  is a local maximum by the Second Derivatives Test.



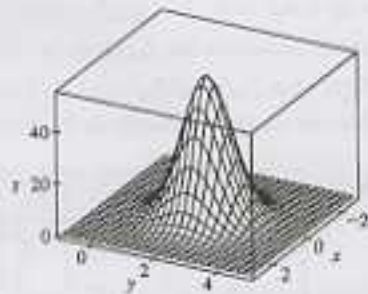
6.  $f(x, y) = x^3y + 12x^2 - 8y \Rightarrow f_x = 3x^2y + 24x, f_y = x^3 - 8,$   
 $f_{xx} = 6xy + 24, f_{xy} = 3x^2, f_{yy} = 0.$  Then  $f_y = 0$  implies  $x = 2,$   
 and substitution into  $f_x = 0$  gives  $12y + 48 = 0 \Rightarrow y = -4.$   
 Thus, the only critical point is  $(2, -4).$   
 $D(2, -4) = (-24)(0) - 12^2 = -144 < 0,$  so  $(2, -4)$  is a saddle point.



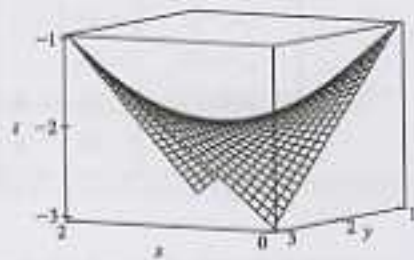
7.  $f(x, y) = x^2 + y^2 + x^2y + 4 \Rightarrow f_x = 2x + 2xy, f_y = 2y + x^2,$   
 $f_{xx} = 2 + 2y, f_{yy} = 2, f_{xy} = 2x.$  Then  $f_y = 0$  implies  $y = -\frac{1}{2}x^2,$   
 substituting into  $f_x = 0$  gives  $2x - x^3 = 0$  so  $x = 0$  or  $x = \pm\sqrt{2}.$   
 Thus the critical points are  $(0, 0), (\sqrt{2}, -1)$  and  $(-\sqrt{2}, -1).$  Now  
 $D(0, 0) = 4, D(\sqrt{2}, -1) = -8 = D(-\sqrt{2}, -1), f_{xx}(0, 0) = 2,$   
 $f_{xx}(\pm\sqrt{2}, -1) = 0.$   
 Thus  $f(0, 0) = 4$  is a local minimum and  $(\pm\sqrt{2}, -1)$  are saddle points.



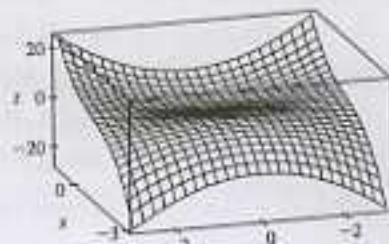
8.  $f(x, y) = e^{4y-x^2-y^2} \Rightarrow f_x = -2xe^{4y-x^2-y^2},$   
 $f_y = (4-2y)e^{4y-x^2-y^2}, f_{xx} = (4x^2-2)e^{4y-x^2-y^2},$   
 $f_{xy} = -2x(4-2y)e^{4y-x^2-y^2},$   
 $f_{yy} = (4y^2-16y+14)e^{4y-x^2-y^2}.$  Then  $f_x = 0$  and  $f_y = 0$   
 implies  $x = 0$  and  $y = 2,$  so the only critical point is  $(0, 2).$   
 $D(0, 2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$  and  
 $f_{xx}(0, 2) = -2e^4 < 0,$  so  $f(0, 2) = e^4$  is a local maximum.



9.  $f(x, y) = xy - 2x - y \Rightarrow f_x = y - 2, f_y = x - 1,$   
 $f_{xx} = f_{yy} = 0, f_{xy} = 1$  and the only critical point is  $(1, 2).$  Now  
 $D(1, 2) = -1,$  so  $(1, 2)$  is a saddle point and  $f$  has no local maximum or minimum.



10.  $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2 \Rightarrow f_x = 6x^2 + y^2 + 10x,$   
 $f_y = 2xy + 2y, f_{xx} = 12x + 10, f_{yy} = 2x + 2, f_{xy} = 2y.$  Then  
 $f_x = 0$  implies  $y = 0$  or  $x = -1.$  Substituting into  $f_y = 0$  gives the  
 critical points  $(0, 0), (-\frac{5}{3}, 0), (-1, \pm 2).$  Now  $D(0, 0) = 20 > 0$   
 and  $f_{xx}(0, 0) = 10 > 0,$  so  $f(0, 0) = 0$  is a local minimum. Also  
 $f_{xx}(-\frac{5}{3}, 0) < 0, D(-\frac{5}{3}, 0) > 0,$  and  $D(-1, \pm 2) < 0.$  Hence  
 $f(-\frac{5}{3}, 0) = \frac{125}{27}$  is a local maximum while  $(-1, \pm 2)$  are saddle



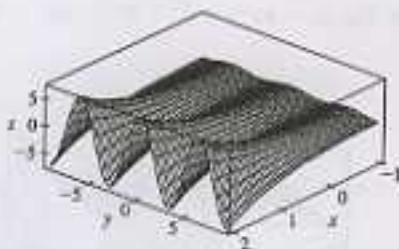
11.  $f(x, y) = \sin(\frac{y}{2})$   
 $f_x = 0$   
 $f_y = \cos(\frac{y}{2})$

12.  $f(x, y) = 2x - y^2$   
 $f_x = 2$   
 $f_y = -2y$   
 or  $y^2 = 0$   
 implies  
 $2x - 0 = 0$   
 $x = 0$   
 $(-1, 1)$   
 $f(±1, 1)$

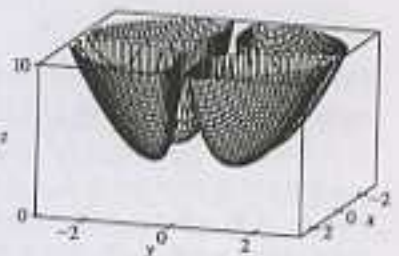
13.  $f(x, y) = y^2 + x^2$   
 $f_{xx} = 2$   
 $f_{yy} = 2$   
 $f_{xy} = 0$   
 for each  
 an inter  
 is a sa

14.  $f(x, y) = x^2 + y^2$   
 $f_x = 2x$   
 $f_y = 2y$   
 $f_{xx} = 2$   
 $f_{yy} = 2$   
 $f_{xy} = 0$   
 $x = 2$   
 critical  
 Now  $D$

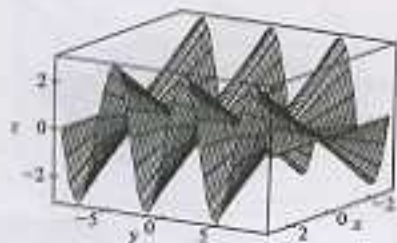
11.  $f(x, y) = e^x \cos y \Rightarrow f_x = e^x \cos y, f_y = -e^x \sin y$ . Now  $f_x = 0$  implies  $\cos y = 0$  or  $y = \frac{\pi}{2} + n\pi$  for  $n$  an integer. But  $\sin(\frac{\pi}{2} + n\pi) \neq 0$ , so there are no critical points.



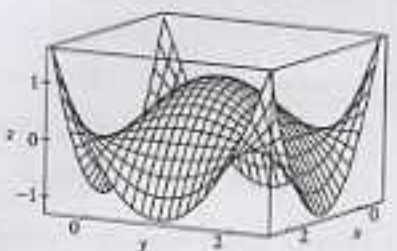
12.  $f(x, y) = x^2 + y^2 + \frac{1}{x^2 y^2} \Rightarrow f_x = 2x - 2x^{-3}y^{-2},$   
 $f_y = 2y - 2x^{-2}y^{-3}, f_{xx} = 2 + 6x^{-4}y^{-2}, f_{yy} = 2 + 6x^{-2}y^{-4},$   
 $f_{xy} = 4x^{-3}y^{-3}$ . Then  $f_x = 0$  implies  $2x^4 y^2 - 2 = 0$  or  $x^4 y^2 = 1$   
 or  $y^2 = x^{-4}$ . Note that neither  $x$  nor  $y$  can be zero. Now  $f_y = 0$   
 implies  $2x^2 y^4 - 2 = 0$ , and with  $y^2 = x^{-4}$  this implies  
 $2x^{-6} - 2 = 0$  or  $x^6 = 1$ . Thus  $x = \pm 1$  and if  $x = 1, y = \pm 1$ ; if  
 $x = -1, y = \pm 1$ . So the critical points are  $(1, 1), (1, -1),$   
 $(-1, 1)$  and  $(-1, -1)$ . Now  $D(\pm 1, \pm 1) = D(\pm 1, \mp 1) = 64 - 16 > 0$  and  $f_{xx} > 0$  always, so  
 $f(\pm 1, \pm 1) = f(\pm 1, \mp 1) = 3$  are local minima.



13.  $f(x, y) = x \sin y \Rightarrow f_x = \sin y, f_y = x \cos y, f_{xx} = 0,$   
 $f_{yy} = -x \sin y$  and  $f_{xy} = \cos y$ . Then  $f_x = 0$  if and only if  
 $y = n\pi, n$  an integer, and substituting into  $f_y = 0$  requires  $x = 0$   
 for each of these  $y$ -values. Thus the critical points are  $(0, n\pi), n$   
 an integer. But  $D(0, n\pi) = -\cos^2(n\pi) < 0$  so each critical point  
 is a saddle point.



14.  $f(x, y) = (2x - x^2)(2y - y^2) \Rightarrow f_x = (2 - 2x)(2y - y^2),$   
 $f_y = (2x - x^2)(2 - 2y), f_{xx} = -2(2y - y^2),$   
 $f_{yy} = -2(2x - x^2)$  and  $f_{xy} = (2 - 2x)(2 - 2y)$ . Then  
 $f_x = 0$  implies  $x = 1$  or  $y = 0$  or  $y = 2$  and when  $x = 1,$   
 $f_y = 0$  implies  $y = 1$ , when  $y = 0, f_y = 0$  implies  $x = 0$  or  
 $x = 2$  and when  $y = 2, f_y = 0$  implies  $x = 0$  or  $x = 2$ . Thus the  
 critical points are  $(1, 1), (0, 0), (2, 0), (0, 2)$  and  $(2, 2)$ .  
 Now  $D(0, 0) = D(2, 0) = D(0, 2) = D(2, 2) = -16$  so these critical points are saddle points, and  $D(1, 1) = 4$   
 with  $f_{xx}(1, 1) = -2$ , so  $f(1, 1) = 1$  is a local maximum.



29.  $f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2 \Rightarrow f_x(x, y) = -2(x^2 - 1)(2x) - 2(x^2y - x - 1)(2xy - 1)$  and  $f_y(x, y) = -2(x^2y - x - 1)x^2$ . Setting  $f_y(x, y) = 0$  gives either  $x = 0$  or  $x^2y - x - 1 = 0$ . There are no critical points for  $x = 0$ , since  $f_x(0, y) = -2$ , so we set  $x^2y - x - 1 = 0 \Rightarrow y = \frac{x+1}{x^2}$  ( $x \neq 0$ ), so

$$f_x\left(x, \frac{x+1}{x^2}\right) = -2(x^2 - 1)(2x) - 2\left(x^2 \frac{x+1}{x^2} - x - 1\right)\left(2x \frac{x+1}{x^2} - 1\right) = -4x(x^2 - 1).$$

$f_x(x, y) = f_y(x, y) = 0$  at the points  $(1, 2)$  and  $(-1, 0)$ .

To classify these critical points, we calculate

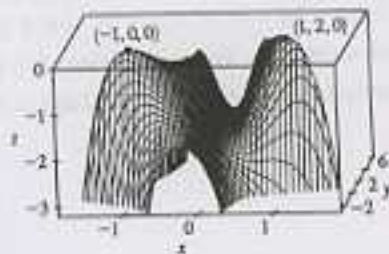
$$f_{xx}(x, y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2,$$

$f_{yy}(x, y) = -2x^4$ , and  $f_{xy}(x, y) = -8x^2y + 6x^2 + 4x$ . In order to use the Second Derivatives Test we calculate

$$D(-1, 0) = f_{xx}(-1, 0)f_{yy}(-1, 0) - [f_{xy}(-1, 0)]^2 \\ = 16 > 0,$$

$$f_{xx}(-1, 0) = -10 < 0, D(1, 2) = 16 > 0, \text{ and}$$

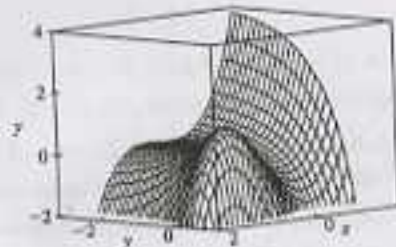
$f_{xx}(1, 2) = -26 < 0$ , so both  $(-1, 0)$  and  $(1, 2)$  give local maxima.



30.  $f(x, y) = 3xe^y - x^3 - e^{3y}$  is differentiable everywhere, so the requirement for critical points is that (1)  $f_x = 3e^y - 3x^2 = 0$  and (2)  $f_y = 3xe^y - 3e^{3y} = 0$ . From (1) we obtain  $e^y = x^2$ , and then (2) gives  $3x^3 - 3x^0 = 0 \Rightarrow x = 1$  or  $0$ , but only  $x = 1$  is valid, since  $x = 0$  makes (1) impossible. So substituting  $x = 1$  into (1) gives  $y = 0$ , and the only critical point is  $(1, 0)$ .

The Second Derivatives Test shows that this gives a local maximum, since

$D(1, 0) = [-6x(3xe^y - 9e^{3y}) - (3e^y)^2]_{(1,0)} = 27 > 0$  and  $f_{xx}(1, 0) = [-6x]_{(1,0)} = -6 < 0$ . But  $f(1, 0) = 1$  is not an absolute maximum because, for instance,  $f(-3, 0) = 17$ . This can also be seen from the graph.



31. Let  $d$  be the distance from  $(2, 1, -1)$  to any point  $(x, y, z)$  on the plane  $x + y - z = 1$ , so

$$d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2} \text{ where } z = x + y - 1, \text{ and we minimize}$$

$$d^2 = f(x, y) = (x-2)^2 + (y-1)^2 + (x+y)^2. \text{ Then } f_x(x, y) = 2(x-2) + 2(x+y) = 4x + 2y - 4,$$

$f_y(x, y) = 2(y-1) + 2(x+y) = 2x + 4y - 2$ . Solving  $4x + 2y - 4 = 0$  and  $2x + 4y - 2 = 0$  simultaneously gives  $x = 1, y = 0$ . An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for  $x = 1, y = 0$  for which

$$d = \sqrt{(1-2)^2 + (0-1)^2 + (1+0)^2} = \sqrt{3}.$$

42. The cost equals  $5xy + 2(xz + yz)$  and  $xyz = V$ , so

$$C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1}). \text{ Then } C'_x = 5y - 2Vx^{-2}, C'_y = 5x - 2Vy^{-2},$$

$f_x = 0$  implies  $y = 2V/(5x^2)$ ,  $f_y = 0$  implies  $x = \sqrt[3]{\frac{2}{5}V} = y$ . Thus the dimensions of the box which minimize the cost are  $x = y = \sqrt[3]{\frac{2}{5}V}$  units,  $z = V^{1/3}/(\frac{5}{2})^{2/3}$ .

43. Let the dimensions be  $x$ ,  $y$  and  $z$ , then minimize  $xy + 2(xz + yz)$  if  $xyz = 32,000 \text{ m}^3$ . Then

$$f(x, y) = xy + [64,000(x + y)/xy] = xy + 64,000(x^{-1} + y^{-1}), f_x = y - 64,000x^{-2}, f_y = x - 64,000y^{-2}.$$

And  $f_x = 0$  implies  $y = 64,000/x^2$ ; substituting into  $f_y = 0$  implies  $x^3 = 64,000$  or  $x = 40$  and then  $y = 40$ .

Now  $D(x, y) = [(2)(64,000)]^2 x^{-3} y^{-3} - 1 > 0$  for  $(40, 40)$  and  $f_{xx}(40, 40) > 0$  so this is indeed a minimum.

Thus the dimensions of the box are  $x = y = 40 \text{ cm}$ ,  $z = 20 \text{ cm}$ .

44. Since  $p + q + r = 1$  we can substitute  $p = 1 - r - q$  into  $P$  giving

$P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$ . Since  $p$ ,  $q$  and  $r$  represent proportions and  $p + q + r = 1$ , we know  $q \geq 0$ ,  $r \geq 0$ , and  $q + r \leq 1$ . Thus, we want to find the absolute maximum of the continuous function  $P(q, r)$  on the closed set  $D$  enclosed by the lines  $q = 0$ ,  $r = 0$ , and

$q + r = 1$ . To find any critical points, we set the partial derivatives equal to zero:  $P_q(q, r) = 2 - 4q - 2r = 0$  and  $P_r(q, r) = 2 - 4r - 2q = 0$ . The first equation gives  $r = 1 - 2q$ , and substituting into the second equation we have  $2 - 4(1 - 2q) - 2q = 0 \Rightarrow q = \frac{1}{3}$ . Then we have one critical point,  $(\frac{1}{3}, \frac{1}{3})$ , where  $P(\frac{1}{3}, \frac{1}{3}) = \frac{2}{3}$ . Next we find

the maximum values of  $P$  on the boundary of  $D$  which consists of three line segments. For the segment given by  $r = 0$ ,  $0 \leq q \leq 1$ ,  $P(q, r) = P(q, 0) = 2q - 2q^2$ ,  $0 \leq q \leq 1$ . This represents a parabola with maximum value  $P(\frac{1}{2}, 0) = \frac{1}{2}$ . On the segment  $q = 0$ ,  $0 \leq r \leq 1$  we have  $P(0, r) = 2r - 2r^2$ ,  $0 \leq r \leq 1$ . This represents a parabola with maximum value  $P(0, \frac{1}{2}) = \frac{1}{2}$ . Finally, on the segment  $q + r = 1$ ,  $0 \leq q \leq 1$ ,

$P(q, r) = P(q, 1 - q) = 2q - 2q^2$ ,  $0 \leq q \leq 1$  which has a maximum value of  $P(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ . Comparing these values with the value of  $P$  at the critical point, we see that the absolute maximum value of  $P(q, r)$  on  $D$  is  $\frac{2}{3}$ .

45. Note that here the variables are  $m$  and  $b$ , and  $f(m, b) = \sum_{i=1}^n [y_i - (mx_i + b)]^2$ . Then

$$f_m = \sum_{i=1}^n -2x_i [y_i - (mx_i + b)] = 0 \text{ implies } \sum_{i=1}^n (x_i y_i - mx_i^2 - bx_i) = 0 \text{ or } \sum_{i=1}^n x_i y_i = m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i$$

$$\text{and } f_b = \sum_{i=1}^n -2[y_i - (mx_i + b)] = 0 \text{ implies } \sum_{i=1}^n y_i = m \sum_{i=1}^n x_i + \sum_{i=1}^n b = m \left( \sum_{i=1}^n x_i \right) + nb. \text{ Thus we have}$$

the two desired equations. Now  $f_{mm} = \sum_{i=1}^n -2x_i^2$ ,  $f_{bb} = \sum_{i=1}^n -2 = -2n$  and  $f_{mb} = \sum_{i=1}^n -2x_i$ . And  $f_{mm}(m, b) > 0$

always and  $D(m, b) = 4n \left( \sum_{i=1}^n x_i^2 \right) - 4 \left( \sum_{i=1}^n x_i \right)^2 = 4 \left[ n \left( \sum_{i=1}^n x_i^2 \right) - \left( \sum_{i=1}^n x_i \right)^2 \right] > 0$  always so the

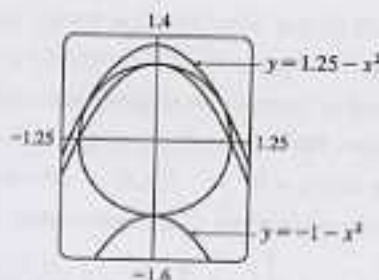
solutions of these two equations do indeed minimize  $\sum_{i=1}^n d_i^2$ .



## Lagrange Multipliers . . . . .

1. At the extreme values of  $f$ , the level curves of  $f$  just touch the curve  $g(x, y) = 8$  with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve  $f(x, y) = c$  with the largest value of  $c$  which still intersects the curve  $g(x, y) = 8$  is approximately  $c = 59$ , and the smallest value of  $c$  corresponding to a level curve which intersects  $g(x, y) = 8$  appears to be  $c = 30$ . Thus we estimate the maximum value of  $f$  subject to the constraint  $g(x, y) = 8$  to be about 59 and the minimum to be 30.

2. (a) The values  $c = \pm 1$  and  $c = 1.25$  seem to give curves which are tangent to the circle. These values represent possible extreme values of the function  $x^2 + y$  subject to the constraint  $x^2 + y^2 = 1$ .



- (b)  $\nabla f = (2x, 1)$ ,  $\lambda \nabla g = (2\lambda x, 2\lambda y)$ . So  $2x = 2\lambda x \Rightarrow$  either  $\lambda = 1$  or  $x = 0$ . If  $\lambda = 1$ , then  $y = \frac{1}{2}$  and so  $x = \pm \frac{\sqrt{3}}{2}$  (from the constraint). If  $x = 0$ , then  $y = \pm 1$ .

Therefore  $f$  has possible extreme values at the

points  $(0, \pm 1)$  and  $(\pm \frac{\sqrt{3}}{2}, \frac{1}{2})$ . We calculate  $f(\pm \frac{\sqrt{3}}{2}, \frac{1}{2}) = \frac{5}{4}$  (the maximum value),  $f(0, 1) = 1$ , and  $f(0, -1) = -1$  (the minimum value). These are our answers from (a).

3.  $f(x, y) = x^2 - y^2$ ,  $g(x, y) = x^2 + y^2 = 1 \Rightarrow \nabla f = (2x, -2y)$ ,  $\lambda \nabla g = (2\lambda x, 2\lambda y)$ . Then  $2x = 2\lambda x$  implies  $x = 0$  or  $\lambda = 1$ . If  $x = 0$ , then  $x^2 + y^2 = 1$  implies  $y = \pm 1$  and if  $\lambda = 1$ , then  $-2y = 2\lambda y$  implies  $y = 0$  and thus  $x = \pm 1$ . Thus the possible points for the extreme values of  $f$  are  $(\pm 1, 0)$ ,  $(0, \pm 1)$ . But  $f(\pm 1, 0) = 1$  while  $f(0, \pm 1) = -1$  so the maximum value of  $f$  on  $x^2 + y^2 = 1$  is  $f(\pm 1, 0) = 1$  and the minimum value is  $f(0, \pm 1) = -1$ .
4.  $f(x, y) = 4x + 6y$ ,  $g(x, y) = x^2 + y^2 = 13 \Rightarrow \nabla f = (4, 6)$ ,  $\lambda \nabla g = (2\lambda x, 2\lambda y)$ . Then  $2\lambda x = 4$  and  $2\lambda y = 6$  imply  $x = \frac{2}{\lambda}$  and  $y = \frac{3}{\lambda}$ . But  $13 = x^2 + y^2 = (\frac{2}{\lambda})^2 + (\frac{3}{\lambda})^2 \Rightarrow 13 = \frac{13}{\lambda^2} \Rightarrow \lambda = \pm 1$ , so  $f$  has possible extreme values at the points  $(2, 3)$ ,  $(-2, -3)$ . We compute  $f(2, 3) = 26$  and  $f(-2, -3) = -26$ , so the maximum value of  $f$  on  $x^2 + y^2 = 13$  is  $f(2, 3) = 26$  and the minimum value is  $f(-2, -3) = -26$ .
5.  $f(x, y) = x^2 y$ ,  $g(x, y) = x^2 + 2y^2 = 6 \Rightarrow \nabla f = (2xy, x^2)$ ,  $\lambda \nabla g = (2\lambda x, 4\lambda y)$ . Then  $2xy = 2\lambda x$  implies  $x = 0$  or  $\lambda = y$ . If  $x = 0$ , then  $x^2 = 4\lambda y$  implies  $\lambda = 0$  or  $y = 0$ . However, if  $y = 0$  then  $g(x, y) = 0$ , a contradiction. So  $\lambda = 0$  and then  $g(x, y) = 6 \Rightarrow y = \pm \sqrt{3}$ . If  $\lambda = y$ , then  $x^2 = 4\lambda y$  implies  $x^2 = 4y^2$ , and so  $g(x, y) = 6 \Rightarrow 4y^2 + 2y^2 = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ . Thus  $f$  has possible extreme values at the points  $(0, \pm \sqrt{3})$ ,  $(\pm 2, 1)$ , and  $(\pm 2, -1)$ . After evaluating  $f$  at these points, we find the maximum value to be  $f(\pm 2, 1) = 4$  and the minimum to be  $f(\pm 2, -1) = -4$ .

6.  $f(x, y) = x^2 + y^2$ ,  $g(x, y) = x^4 + y^4 = 1 \Rightarrow \nabla f = \langle 2x, 2y \rangle$ ,  $\lambda \nabla g = \langle 4\lambda x^3, 4\lambda y^3 \rangle$ . Then  $x = 2\lambda x^3$  implies  $x = 0$  or  $\lambda = \frac{1}{2x^2}$ . If  $x = 0$ , then  $x^4 + y^4 = 1$  implies  $y = \pm 1$ . But  $y = 2\lambda y^3$  implies  $y = 0$  so  $x = \pm 1$  or  $\lambda = \frac{1}{2y^2}$  and  $x^2 = y^2$  and  $2x^4 = 1$  so  $x = \pm \frac{1}{\sqrt[4]{2}}$ . Hence the possible points are  $(0, \pm 1)$ ,  $(\pm 1, 0)$ ,  $(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}})$ , with the maximum value of  $f$  on  $x^4 + y^4 = 1$  being  $f(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}) = \frac{2}{\sqrt[4]{2}} = \sqrt{2}$  and the minimum value being  $f(0, \pm 1) = f(\pm 1, 0) = 1$ .
7.  $f(x, y, z) = 2x + 6y + 10z$ ,  $g(x, y, z) = x^2 + y^2 + z^2 = 35 \Rightarrow \nabla f = \langle 2, 6, 10 \rangle$ ,  $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$ . Then  $2\lambda x = 2$ ,  $2\lambda y = 6$ ,  $2\lambda z = 10$  imply  $x = \frac{1}{\lambda}$ ,  $y = \frac{3}{\lambda}$ , and  $z = \frac{5}{\lambda}$ . But  $35 = x^2 + y^2 + z^2 = \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 \Rightarrow 35 = \frac{35}{\lambda^2} \Rightarrow \lambda = \pm 1$ , so  $f$  has possible extreme values at the points  $(1, 3, 5)$ ,  $(-1, -3, -5)$ . The maximum value of  $f$  on  $x^2 + y^2 + z^2 = 35$  is  $f(1, 3, 5) = 70$ , and the minimum is  $f(-1, -3, -5) = -70$ .
8.  $f(x, y, z) = 8x - 4z$ ,  $g(x, y, z) = x^2 + 10y^2 + z^2 = 5 \Rightarrow \nabla f = \langle 8, 0, -4 \rangle$ ,  $\lambda \nabla g = \langle 2\lambda x, 20\lambda y, 2\lambda z \rangle$ . Then  $2\lambda x = 8$ ,  $20\lambda y = 0$ ,  $2\lambda z = -4$  imply  $x = \frac{4}{\lambda}$ ,  $y = 0$ , and  $z = -\frac{2}{\lambda}$ . But  $5 = x^2 + 10y^2 + z^2 = \left(\frac{4}{\lambda}\right)^2 + 10(0)^2 + \left(-\frac{2}{\lambda}\right)^2 \Rightarrow 5 = \frac{20}{\lambda^2} \Rightarrow \lambda = \pm 2$ , so  $f$  has possible extreme values at the points  $(2, 0, -1)$ ,  $(-2, 0, 1)$ . The maximum of  $f$  on  $x^2 + 10y^2 + z^2 = 5$  is  $f(2, 0, -1) = 20$ , and the minimum is  $f(-2, 0, 1) = -20$ .
9.  $f(x, y, z) = xyz$ ,  $g(x, y, z) = x^2 + 2y^2 + 3z^2 = 6 \Rightarrow \nabla f = \langle yz, xz, xy \rangle$ ,  $\lambda \nabla g = \langle 2\lambda x, 4\lambda y, 6\lambda z \rangle$ . Then  $\nabla f = \lambda \nabla g$  implies  $\lambda = (yz)/(2x) = (xz)/(4y) = (xy)/(6z)$  or  $x^2 = 2y^2$  and  $x^2 = \frac{2}{3}y^2$ . Thus  $x^2 + 2y^2 + 3z^2 = 6$  implies  $6y^2 = 6$  or  $y = \pm 1$ . Then the possible points are  $(\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}})$ ,  $(\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}})$ ,  $(-\sqrt{2}, \pm 1, \sqrt{\frac{2}{3}})$ ,  $(-\sqrt{2}, \pm 1, -\sqrt{\frac{2}{3}})$ . The maximum value of  $f$  on the ellipsoid is  $\frac{2\sqrt{6}}{3}$ , occurring when all coordinates are positive or exactly two are negative and the minimum is  $-\frac{2\sqrt{6}}{3}$  occurring when 1 or 3 of the coordinates are negative.
10.  $f(x, y, z) = x^2 y^2 z^2$ ,  $g(x, y, z) = x^2 + y^2 + z^2 = 1 \Rightarrow \nabla f = \langle 2xy^2z^2, 2yz^2x^2, 2xz^2y^2 \rangle$ ,  $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$ . Then  $\nabla f = \lambda \nabla g$  implies (1)  $\lambda = y^2 z^2 = x^2 z^2 = x^2 y^2$  and  $\lambda \neq 0$ , or (2)  $\lambda = 0$  and one or two (but not three) of the coordinates are 0. If (1) then  $x^2 = y^2 = z^2 = \frac{1}{3}$ . The minimum value of  $f$  on the sphere occurs in case (2) with a value of 0 and the maximum value is  $\frac{1}{27}$  which arises from all the points from (1), that is, the points  $(\pm \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ ,  $(\pm \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ ,  $(\pm \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ .
11.  $f(x, y, z) = x^2 + y^2 + z^2$ ,  $g(x, y, z) = x^4 + y^4 + z^4 = 1 \Rightarrow \nabla f = \langle 2x, 2y, 2z \rangle$ ,  $\lambda \nabla g = \langle 4\lambda x^3, 4\lambda y^3, 4\lambda z^3 \rangle$ .  
Case 1: If  $x \neq 0$ ,  $y \neq 0$  and  $z \neq 0$ , then  $\nabla f = \lambda \nabla g$  implies  $\lambda = 1/(2x^2) = 1/(2y^2) = 1/(2z^2)$  or  $x^2 = y^2 = z^2$  and  $3x^4 = 1$  or  $x = \pm \frac{1}{\sqrt[4]{3}}$  giving the points  $(\pm \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$ ,  $(\pm \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$ ,  $(\pm \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}})$ ,  $(\pm \frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}}, -\frac{1}{\sqrt[4]{3}})$  all with an  $f$ -value of  $\sqrt{3}$ .

$$21. P(L, K) = bL^\alpha K^{1-\alpha}, g(L, K) = mL + nK = p \Rightarrow \nabla P = (\alpha bL^{\alpha-1}K^{1-\alpha}, (1-\alpha)bL^\alpha K^{-\alpha}).$$

$\lambda \nabla g = (\lambda m, \lambda n)$ . Then  $\alpha b(L/K)^{1-\alpha} = \lambda m$  and  $(1-\alpha)b(L/K)^\alpha = \lambda n$  and  $mL + nK = p$ , so

$$\alpha b(K/L)^{1-\alpha}/m = (1-\alpha)b(L/K)^\alpha/n \text{ or } n\alpha/[m(1-\alpha)] = (L/K)^\alpha(L/K)^{1-\alpha} \text{ or } L = Kn\alpha/[m(1-\alpha)].$$

Substituting into  $mL + nK = p$  gives  $K = (1-\alpha)p/n$  and  $L = \alpha p/m$  for the maximum production.

$$22. C(L, K) = mL + nK, g(L, K) = bL^\alpha K^{1-\alpha} = Q \Rightarrow \nabla C = (m, n),$$

$\lambda \nabla g = (\lambda \alpha bL^{\alpha-1}K^{1-\alpha}, \lambda(1-\alpha)bL^\alpha K^{-\alpha})$ . Then  $\frac{m}{\alpha b} \left(\frac{L}{K}\right)^{1-\alpha} = \frac{n}{(1-\alpha)b} \left(\frac{K}{L}\right)^\alpha$  and  $bL^\alpha K^{1-\alpha} = Q$

$$\Rightarrow \frac{n\alpha}{m(1-\alpha)} = \left(\frac{L}{K}\right)^{1-\alpha} \left(\frac{L}{K}\right)^\alpha \Rightarrow L = \frac{Kn\alpha}{m(1-\alpha)} \text{ and so } b \left[\frac{Kn\alpha}{m(1-\alpha)}\right]^\alpha K^{1-\alpha} = Q. \text{ Hence}$$

$$K = \frac{Q}{b(n\alpha/[m(1-\alpha)])^\alpha} = \frac{Qm^\alpha(1-\alpha)^\alpha}{bn^\alpha\alpha^\alpha} \text{ and } L = \frac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}} = \frac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}} \text{ minimizes cost.}$$

$$23. \text{ Let the sides of the rectangle be } x \text{ and } y. \text{ Then } f(x, y) = xy, g(x, y) = 2x + 2y = p \Rightarrow \nabla f(x, y) = (y, x),$$

$\lambda \nabla g = (2\lambda, 2\lambda)$ . Then  $\lambda = \frac{1}{2}y = \frac{1}{2}x$  implies  $x = y$  and the rectangle with maximum area is a square with side length  $\frac{1}{4}p$ .

$$24. \text{ Let } f(x, y, z) = s(s-x)(s-y)(s-z), g(x, y, z) = x + y + z. \text{ Then}$$

$$\nabla f = (-s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y)), \lambda \nabla g = (\lambda, \lambda, \lambda). \text{ Thus}$$

(1)  $(s-y)(s-z) = (s-x)(s-z)$  and (2)  $(s-x)(s-z) = (s-x)(s-y)$ . (1) implies  $x = y$  while (2) implies  $y = z$ , so  $x = y = z = p/3$  and the triangle with maximum area is equilateral.

$$25. \text{ Let } f(x, y, z) = d^2 = (x-2)^2 + (y-1)^2 + (z+1)^2, \text{ then we want to minimize } f \text{ subject to the constraint}$$

$$g(x, y, z) = x + y - z = 1, \nabla f = \lambda \nabla g \Rightarrow (2(x-2), 2(y-1), 2(z+1)) = \lambda(1, 1, -1), \text{ so}$$

$$x = (\lambda+4)/2, y = (\lambda+2)/2, z = -(\lambda+2)/2. \text{ Substituting into the constraint equation gives}$$

$$\frac{\lambda+4}{2} + \frac{\lambda+2}{2} - \frac{\lambda+2}{2} = 1 \Rightarrow 3\lambda+8=2 \Rightarrow \lambda=-2, \text{ so } x=1, y=0, \text{ and } z=0. \text{ This must}$$

correspond to a minimum, so the shortest distance is  $d = \sqrt{(1-2)^2 + (0-1)^2 + (0+1)^2} = \sqrt{3}$ .

$$26. \text{ Let } f(x, y, z) = d^2 = (x-1)^2 + (y-2)^2 + (z-3)^2, \text{ then we want to minimize } f \text{ subject to the constraint}$$

$$g(x, y, z) = x - y + z = 4, \nabla f = \lambda \nabla g \Rightarrow (2(x-1), 2(y-2), 2(z-3)) = \lambda(1, -1, 1), \text{ so}$$

$$x = (\lambda+2)/2, y = (4-\lambda)/2, z = (\lambda+6)/2. \text{ Substituting into the constraint equation gives}$$

$$\frac{\lambda+2}{2} - \frac{4-\lambda}{2} + \frac{\lambda+6}{2} = 4 \Rightarrow \lambda = \frac{4}{3}, \text{ so } x = \frac{5}{3}, y = \frac{8}{3}, \text{ and } z = \frac{11}{3}. \text{ This must correspond to a minimum,}$$

so the point on the plane closest to the point  $(1, 2, 3)$  is  $(\frac{5}{3}, \frac{8}{3}, \frac{11}{3})$ .

$$27. f(x, y, z) = x^2 + y^2 + z^2, g(x, y, z) = x^2 - xy - 1 = 0 \Rightarrow \nabla f = (2x, 2y, 2z) = \lambda \nabla g = (-\lambda y, -\lambda x, 2\lambda z).$$

Then  $2x = 2\lambda z$  implies  $z = 0$  or  $\lambda = 1$ . If  $z = 0$  then  $g(x, y, z) = 1$  implies  $xy = -1$  or  $x = -1/y$ . Thus

$$2x = -\lambda y \text{ and } 2y = -\lambda x \text{ imply } \lambda = 2/y^2 = 2x^2 \text{ or } y = \pm 1, x = \pm 1. \text{ If } \lambda = 1, \text{ then } 2x = -y \text{ and } 2y = -x$$

imply  $x = y = 0$ , so  $z = \pm 1$ . Hence the possible points are  $(\pm 1, \mp 1, 0)$ ,  $(0, 0, \pm 1)$  and the minimum value of  $f$  is

$f(0, 0, \pm 1) = 1$ , so the points closest to the origin are  $(0, 0, \pm 1)$ .