17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$ for any choice of sample points (x_{ij}^*, y_{ij}^*). But $f(x_{ij}^*, y_{ij}^*) = k$ always and $\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta A = \text{area of } R = (b-a)(d-c)$. Thus, no matter how we choose the sample points, $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A = k\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A = k(b-a)(d-c)$ and so

$$\begin{aligned}\iint_R k\,dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A = \lim_{m,n\to\infty} k\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A\\ &= \lim_{m,n\to\infty} k(b-a)(d-c) = k(b-a)(d-c)\end{aligned}$$

18. On R, $0 \le x+y \le 2 < \pi$ and $\sin\theta \ge 0$ for $0 \le \theta \le \pi$. Thus $f(x,y) = \sin(x+y) \ge 0$ for all $(x,y) \in R$. Since $0 \le \sin(x+y) \le 1$, Property (9) gives $\iint_R 0\,dA \le \iint_R \sin(x+y)\,dA \le \iint_R 1\,dA$, so by Exercise 17 we have $0 \le \iint_R \sin(x+y)\,dA \le 1$.

12.2 Iterated Integrals

1. $\int_0^3 (2x+3x^2y)\,dx = \left[x^2+x^3y\right]_{x=0}^{x=3} = (9+27y)-(0+0) = 9+27y$,

$\int_0^4 (2x+3x^2y)\,dy = \left[2xy+3x^2\frac{y^2}{2}\right]_{y=0}^{y=4} = \left(8x+3x^2\cdot\frac{16}{2}\right)-(0+0) = 8x+24x^2$

2. $\displaystyle\int_0^3 \frac{y}{x+2}\,dx = y\ln|x+2|\Big]_{x=0}^{x=3} = y\ln 5 - y\ln 2 = y\ln\frac{5}{2}$,

$\displaystyle\int_0^4 \frac{y}{x+2}\,dy = \frac{1}{x+2}\left[\frac{y^2}{2}\right]_{y=0}^{y=4} = \frac{1}{x+2}\left(\frac{16}{2}-0\right) = \frac{8}{x+2}$

3. $\int_1^3\int_0^1 (1+4xy)\,dx\,dy = \int_1^3 \left[x+2x^2y\right]_{x=0}^{x=1}dy = \int_1^3 (1+2y)\,dy$
$= \left[y+y^2\right]_1^3 = (3+9)-(1+1) = 10$

4. $\int_2^4\int_{-1}^1 (x^2+y^2)\,dy\,dx = \int_2^4 \left[x^2y+\frac{1}{3}y^3\right]_{y=-1}^{y=1}dx = \int_2^4 \left[\left(x^2+\frac{1}{3}\right)-\left(-x^2-\frac{1}{3}\right)\right]dx$
$= \int_2^4 \left(2x^2+\frac{2}{3}\right)dx = \left[\frac{2}{3}x^3+\frac{2}{3}x\right]_2^4 = \left(\frac{128}{3}+\frac{8}{3}\right)-\left(\frac{16}{3}+\frac{4}{3}\right) = \frac{116}{3}$

5. $\int_0^3\int_0^1 \sqrt{x+y}\,dx\,dy = \int_0^3 \left[\frac{2}{3}(x+y)^{3/2}\right]_{x=0}^{x=1}dy = \frac{2}{3}\int_0^3 \left[(1+y)^{3/2}-y^{3/2}\right]dy$
$= \frac{2}{3}\left[\frac{2}{5}(1+y)^{5/2}-\frac{2}{5}y^{5/2}\right]_0^3 = \frac{4}{15}\left[32-3^{5/2}-1\right] = \frac{4}{15}(31-9\sqrt{3})$

6. $\int_1^4\int_0^2 (x+\sqrt{y})\,dx\,dy = \int_1^4 \left[\frac{1}{2}x^2+x\sqrt{y}\right]_{x=0}^{x=2}dy = \int_1^4 (2+2\sqrt{y})\,dy$
$= \left[2y+2\cdot\frac{2}{3}y^{3/2}\right]_1^4 = \left(8+\frac{4}{3}\cdot 8\right)-\left(2+\frac{4}{3}\right) = \frac{50}{3}$

7. $\displaystyle\int_1^4\int_1^2 \left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx = \int_1^4 \left[x\ln|y|+\frac{1}{x}\cdot\frac{1}{2}y^2\right]_{y=1}^{y=2}dx = \int_1^4 \left(x\ln 2+\frac{3}{2x}\right)dx$
$= \left[\frac{1}{2}x^2\ln 2+\frac{3}{2}\ln|x|\right]_1^4 = 8\ln 2+\frac{3}{2}\ln 4-\frac{1}{2}\ln 2$
$= \frac{15}{2}\ln 2+3\ln 4^{1/2} = \frac{21}{2}\ln 2$

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ll $(x, y) \in R$. Since
xercise 17 we have

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8. $\int_0^{\pi/2}\int_0^{\pi/2}\sin(x+y)\,dy\,dx = \int_0^{\pi/2}\left[-\cos(x+y)\right]_{y=0}^{y=\pi/2}dx = \int_0^{\pi/2}\left[\cos x - \cos\left(x+\frac{\pi}{2}\right)\right]dx$
$= \left[\sin x - \sin\left(x+\frac{\pi}{2}\right)\right]_0^{\pi/2} = (1-0)-(0-1) = 2$

9. $\int_0^{\ln 2}\int_0^{\ln 5} e^{2x-y}\,dx\,dy = \left(\int_0^{\ln 5} e^{2x}\,dx\right)\left(\int_0^{\ln 2} e^{-y}\,dy\right) = \left[\frac{1}{2}e^{2x}\right]_0^{\ln 5}\left[-e^{-y}\right]_0^{\ln 2}$
$= \left(\frac{25}{2}-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right) = 6$

10. $\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx = \int_0^1\left[x\sqrt{x^2+y^2+1}\right]_{y=0}^{y=1}dx = \int_0^1 x\left(\sqrt{x^2+2}-\sqrt{x^2+1}\right)dx$
$= \frac{1}{3}\left[(x^2+2)^{3/2}-(x^2+1)^{3/2}\right]_0^1 = \frac{1}{3}\left[\left(3^{3/2}-2^{3/2}\right)-\left(2^{3/2}-1\right)\right]$
$= \frac{1}{3}\left(3\sqrt{3}-4\sqrt{2}+1\right)$

11. $\iint_R (6x^2y^3-5y^4)\,dA = \int_0^3\int_0^1 (6x^2y^3-5y^4)\,dy\,dx = \int_0^3\left[\frac{3}{2}x^2y^4-y^5\right]_{y=0}^{y=1}dx$
$= \int_0^3\left(\frac{3}{2}x^2-1\right)dx = \left[\frac{1}{2}x^3-x\right]_0^3 = \frac{27}{2}-3 = \frac{21}{2}$

12. $\iint_R xye^y\,dA = \int_0^2\int_0^1 xye^y\,dy\,dx = \int_0^2 x\,dx\int_0^1 ye^y\,dy = \left[\frac{1}{2}x^2\right]_0^2\left[e^y(y-1)\right]_0^1$ (by integrating by parts)
$= \frac{1}{2}(4-0)(0+e^0) = 2$

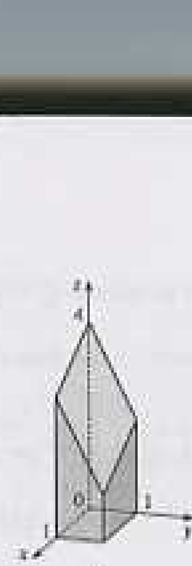
13. $\iint_R \frac{xy^2}{x^2+1}\,dA = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx = \int_0^1 \frac{x}{x^2+1}\,dx\int_{-3}^3 y^2\,dy$
$= \left[\frac{1}{2}\ln(x^2+1)\right]_0^1\left[\frac{1}{3}y^3\right]_{-3}^3 = \frac{1}{2}(\ln 2-\ln 1)\cdot\frac{1}{3}(27+27) = 9\ln 2$

14. $\iint_R \frac{1+x^2}{1+y^2}\,dA = \int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx = \int_0^1 (1+x^2)\,dx\int_0^1 \frac{1}{1+y^2}\,dy$
$= \left[x+\frac{1}{3}x^3\right]_0^1\left[\tan^{-1}y\right]_0^1 = \left(1+\frac{1}{3}-0\right)\left(\frac{\pi}{4}-0\right) = \frac{\pi}{3}$

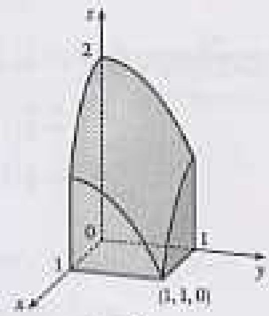
15. $\int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\,dy\,dx$
$= \int_0^{\pi/6}\left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3}dx = \int_0^{\pi/6}\left[x\cos x - x\cos\left(x+\frac{\pi}{3}\right)\right]dx$
$= x\left[\sin x - \sin\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6}\left[\sin x - \sin\left(x+\frac{\pi}{3}\right)\right]dx$
(by integrating by parts separately for each term)
$= \frac{\pi}{6}\left[\frac{1}{2}-1\right] - \left[-\cos x + \cos\left(x+\frac{\pi}{3}\right)\right]_0^{\pi/6} = -\frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2}+0-\left(-1+\frac{1}{2}\right)\right]$
$= \frac{\sqrt{3}-1}{2} - \frac{\pi}{12}$

16. $\int_0^1\int_0^1 xe^{xy}\,dy\,dx = \int_0^1\left[e^{xy}\right]_{y=0}^{y=1}dx = \int_0^1 (e^x-1)\,dx = \left[e^x-x\right]_0^1 = e-2$

17. $z = f(x,y) = 4 - x - 2y \ge 0$ for $0 \le x \le 1$ and $0 \le y \le 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0,1] \times [0,1]$.



18. $z = 2 - x^2 - y^2 \ge 0$ for $0 \le x \le 1$ and $0 \le y \le 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0,1] \times [0,1]$.



19. $V = \iint_R (2x + 5y + 1)\, dA = \int_1^4 \int_{-1}^0 (2x + 5y + 1)\, dx\, dy = \int_1^4 \left[x^2 + 5xy + x\right]_{x=-1}^{x=0} dy$
$= \int_1^4 5y\, dy = \frac{5}{2} y^2\Big]_1^4 = \frac{75}{2}$

20. $V = \iint_R (x^2 + y^2)\, dA = \int_{-3}^3 \int_{-2}^2 (x^2 + y^2)\, dx\, dy = \int_{-3}^3 \left[\frac{1}{3}x^3 + y^2 x\right]_{x=-2}^{x=2} dy$
$= \int_{-3}^3 \left[\frac{16}{3} + 4y^2\right] dy = \left[\frac{16}{3}y + \frac{4}{3}y^3\right]_{-3}^3 = 2(16 + 36) = 104$

21. $V = \int_{-2}^2 \int_{-1}^1 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\, dy = 4 \int_0^2 \int_0^1 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\, dy$
$= 4 \int_0^2 \left[x - \frac{1}{12}x^3 - \frac{1}{9}y^2 x\right]_{x=0}^{x=1} dy = 4 \int_0^2 \left(\frac{11}{12} - \frac{1}{9}y^2\right) dy = 4\left[\frac{11}{12}y - \frac{1}{27}y^3\right]_0^2 = 4 \cdot \frac{83}{54} = \frac{166}{27}$

22. $V = \int_1^3 \int_{-1}^1 (y^2 - x^2)\, dx\, dy = 2 \int_1^3 \int_0^1 (y^2 - x^2)\, dx\, dy = 2 \int_1^3 \left[y^2 x - \frac{1}{3}x^3\right]_{x=0}^{x=1} dy$
$= 2 \int_1^3 \left(y^2 - \frac{1}{3}\right) dy = \frac{2}{3}\left[y^3 - y\right]_1^3 = 16$

23. Here we need the volume of the solid lying under the surface $z = x\sqrt{x^2 + y}$ and above the square $R = [0,1] \times [0,1]$ in the xy-plane.

$$V = \int_0^1 \int_0^1 x\sqrt{x^2 + y}\, dx\, dy = \int_0^1 \frac{1}{3}\left[(x^2 + y)^{3/2}\right]_{x=0}^{x=1} dy = \frac{1}{3}\int_0^1 \left[(1+y)^{3/2} - y^{3/2}\right] dy$$
$$= \frac{1}{3} \cdot \frac{2}{5}\left[(1+y)^{5/2} - y^{5/2}\right]_0^1 = \frac{4}{15}\left(2\sqrt{2} - 1\right)$$

24. Here we need the volume of the solid lying under the surface $z = 1 + (x-1)^2 + 4y^2$ and above the rectangle $R = [0,3] \times [0,2]$ in the xy-plane.

$$V = \int_0^3 \int_0^2 \left[1 + (x-1)^2 + 4y^2\right] dy\, dx = \int_0^3 \left[y + (x-1)^2 y + \frac{4}{3}y^3\right]_{y=0}^{y=2} dx$$
$$= \int_0^3 \left[2 + 2(x-1)^2 + \frac{32}{3}\right] dx = \left[\frac{38}{3}x + \frac{2}{3}(x-1)^3\right]_0^3 = 44$$

25. In the first octant, $z \ge 0 \quad \Rightarrow \quad y \le 3$, so
$V = \int_0^3 \int_0^2 (9 - y^2)\, dx\, dy = \int_0^3 \left[9x - y^2 x\right]_{x=0}^{x=2} dy = \int_0^3 (18 - 2y^2)\, dy = \left[18y - \frac{2}{3}y^3\right]_0^3 = 36$

31. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$.
To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

32. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy}, we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1:
$g_x = \dfrac{d}{dx} g(x,y) = \dfrac{d}{dx} \displaystyle\int_a^x \left(\int_c^y f(s,t)\,dt \right) ds = \int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again: $g_{xy} = \dfrac{d}{dy} \displaystyle\int_c^y f(x,t)\,dt = f(x,y)$.
To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x \int_c^y f(s,t)\,dt\,ds = \int_c^y \int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x,y)$. So $g_{xy} = g_{yx} = f(x,y)$.

12.3 Double Integrals over General Regions

1. $\int_0^1 \int_0^{x^2} (x+2y)\,dy\,dx = \int_0^1 \left[xy + y^2\right]_{y=0}^{y=x^2} dx = \int_0^1 \left[x(x^2) + (x^2)^2 - 0 - 0\right] dx$
$= \int_0^1 (x^3 + x^4)\,dx = \left[\frac{1}{4}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{9}{20}$

2. $\int_1^2 \int_y^2 xy\,dx\,dy = \int_1^2 \left[\frac{1}{2}x^2 y\right]_{x=y}^{x=2} dy = \int_1^2 \frac{1}{2}y(4-y^2)\,dy = \frac{1}{2}\int_1^2 (4y - y^3)\,dy$
$= \frac{1}{2}\left[2y^2 - \frac{1}{4}y^4\right]_1^2 = \frac{1}{2}\left(8 - 4 - 2 + \frac{1}{4}\right) = \frac{9}{8}$

3. $\int_0^1 \int_y^{e^y} \sqrt{x}\,dx\,dy = \int_0^1 \left[\frac{2}{3}x^{3/2}\right]_{x=y}^{x=e^y} dy = \frac{2}{3}\int_0^1 \left(e^{3y/2} - y^{3/2}\right) dy = \frac{2}{3}\left[\frac{2}{3}e^{3y/2} - \frac{2}{5}y^{5/2}\right]_0^1$
$= \frac{2}{3}\left(\frac{2}{3}e^{3/2} - \frac{2}{5} - \frac{2}{3}e^0 + 0\right) = \frac{4}{9}e^{3/2} - \frac{32}{45}$

4. $\int_0^1 \int_x^{2-x} (x^2 - y)\,dy\,dx = \int_0^1 \left[x^2 y - \frac{1}{2}y^2\right]_{y=x}^{y=2-x} dx = \int_0^1 \left[x^2(2-x) - \frac{1}{2}(2-x)^2 - x^2(x) + \frac{1}{2}x^2\right] dx$
$= \int_0^1 (-2x^3 + 2x^2 + 2x - 2)\,dx = \left[-\frac{1}{2}x^4 + \frac{2}{3}x^3 + x^2 - 2x\right]_0^1 = -\frac{5}{6}$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta = \int_0^{\pi/2} \left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta} d\theta = \int_0^{\pi/2} (\cos\theta)\,e^{\sin\theta}\,d\theta = e^{\sin\theta}\Big]_0^{\pi/2}$
$= e^{\sin(\pi/2)} - e^0 = e - 1$

6. $\int_0^1 \int_0^v \sqrt{1-v^2}\,du\,dv = \int_0^1 \left[u\sqrt{1-v^2}\right]_{u=0}^{u=v} dv = \int_0^1 v\sqrt{1-v^2}\,dv = -\frac{1}{3}\left(1-v^2\right)^{3/2}\Big]_0^1$
$= -\frac{1}{3}(0-1) = \frac{1}{3}$

7. $\iint_D x^3 y^2\,dA = \int_0^2 \int_{-x}^{x} x^3 y^2\,dy\,dx = \int_0^2 \left[\frac{1}{3}x^3 y^3\right]_{y=-x}^{y=x} dx = \frac{1}{3}\int_0^2 2x^6\,dx$
$= \frac{2}{3}\left[\frac{1}{7}x^7\right]_0^2 = \frac{2}{21}\left[2^7 - 0\right] = \frac{256}{21}$

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$x^2]\,dx$

8. $\iint_D \frac{4y}{x^3+2}\,dA = \int_1^2\int_0^{2x}\frac{4y}{x^3+2}\,dy\,dx = \int_1^2\left[\frac{2y^2}{x^3+2}\right]_{y=0}^{y=2x}dx = \int_1^2\frac{8x^2}{x^3+2}\,dx$

$= \frac{8}{3}\ln|x^3+2|\Big]_1^2 = \frac{8}{3}(\ln 10-\ln 3) = \frac{8}{3}\ln\frac{10}{3}$

9. $\int_0^1\int_0^{\sqrt{x}}\frac{2y}{x^2+1}\,dy\,dx = \int_0^1\left[\frac{y^2}{x^2+1}\right]_{y=0}^{y=\sqrt{x}}dx = \int_0^1\frac{x}{x^2+1}\,dx$

$= \frac12\ln|x^2+1|\Big]_0^1 = \frac12(\ln 2-\ln 1) = \frac12\ln 2$

10. $\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1\left[xe^{y^2}\right]_{x=0}^{x=y}dy = \int_0^1 ye^{y^2}\,dy = \frac12 e^{y^2}\Big]_0^1 = \frac12(e-1)$

11. $\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1[x\sin y]_{y=0}^{y=x^2}\,dx = \int_0^1 x\sin x^2\,dx = -\frac12\cos x^2\Big]_0^1 = \frac12(1-\cos 1)$

12. $\int_0^1\int_0^y x\sqrt{y^2-x^2}\,dx\,dy = \int_0^1\left[-\frac13(y^2-x^2)^{3/2}\right]_{x=0}^{x=y}dy = \frac13\int_0^1 y^3\,dy = \frac13\cdot\frac14 y^4\Big]_0^1 = \frac1{12}$

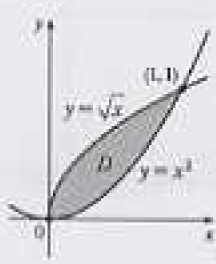
13.

$\int_1^2\int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2\left[xy^3\right]_{x=2-y}^{x=2y-1}dy = \int_1^2[(2y-1)-(2-y)]\,y^3\,dy$

$= \int_1^2(3y^4-3y^3)\,dy = \left[\frac35 y^5-\frac34 y^4\right]_1^2$

$= \frac{96}{5}-12-\frac35+\frac34 = \frac{147}{20}$

14.

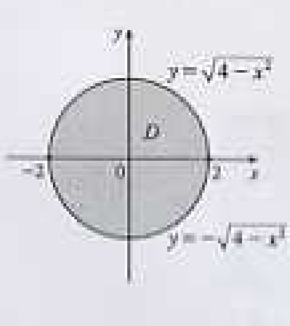


$\int_0^1\int_{x^2}^{\sqrt{x}}(x+y)\,dy\,dx = \int_0^1\left[xy+\frac12y^2\right]_{y=x^2}^{y=\sqrt{x}}dx$

$= \int_0^1\left(x^{3/2}+\frac12x-x^3-\frac12x^4\right)dx$

$= \left[\frac25x^{5/2}+\frac14x^2-\frac14x^4-\frac1{10}x^5\right]_0^1 = \frac{3}{10}$

15.



$\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x-y)\,dy\,dx$

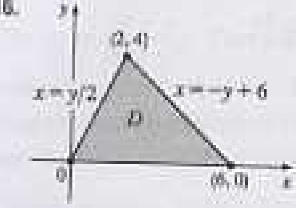
$= \int_{-2}^2\left[2xy-\frac12y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}dx$

$= \int_{-2}^2\left[2x\sqrt{4-x^2}-\frac12(4-x^2)+2x\sqrt{4-x^2}+\frac12(4-x^2)\right]dx$

$= \int_{-2}^2 4x\sqrt{4-x^2}\,dx = -\frac43(4-x^2)^{3/2}\Big]_{-2}^2 = 0$

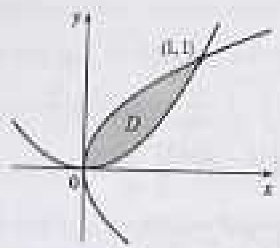
(Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^2 4x\sqrt{4-x^2}\,dx = 0$.)

16.



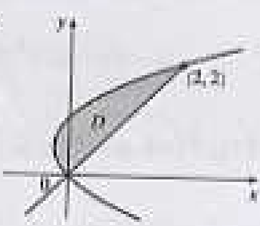
$$\int_0^4 \int_{y/2}^{6-y} ye^x \, dx \, dy = \int_0^4 \left[ye^x\right]_{x=y/2}^{x=6-y} dy$$
$$= \int_0^4 \left(ye^{6-y} - ye^{y/2}\right) dy$$
$$= \left[y\left(-e^{6-y} - 2e^{y/2}\right)\right]_0^4 + \left[-e^{6-y} + 4e^{y/2}\right]_0^4$$
(by integrating by parts separately for each term)
$$= -12e^2 + 3e^2 + e^6 - 4$$
$$= e^6 - 9e^2 - 4$$

17.



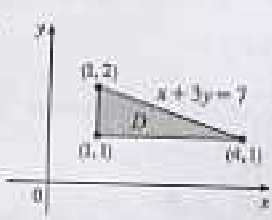
$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} \left(x^2 + y^2\right) dy \, dx = \int_0^1 \left[\left(x^2 y + \frac{y^3}{3}\right)\right]_{y=x^2}^{y=\sqrt{x}} dx$$
$$= \int_0^1 \left(x^{5/2} - x^4 + \tfrac{1}{3}x^{3/2} - \tfrac{1}{3}x^6\right) dx$$
$$= \left[\tfrac{2}{7}x^{7/2} - \tfrac{1}{5}x^5 + \tfrac{2}{15}x^{5/2} - \tfrac{1}{21}x^7\right]_0^1$$
$$= \tfrac{18}{105} = \tfrac{6}{35}$$

18.



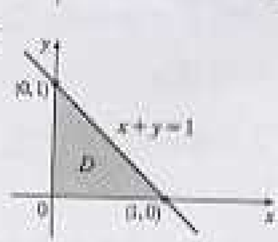
$$V = \int_0^2 \int_{y^2-y}^{y} \left(3x^2 + y^2\right) dx \, dy$$
$$= \int_0^2 \left[x^3 + y^2 x\right]_{x=y^2-y}^{x=y} dy$$
$$= \int_0^2 \left[2y^3 - \left(y^6 - 3y^5 + 4y^4 - 2y^3\right)\right] dy$$
$$= \left[-\frac{y^7}{7} + \frac{y^6}{2} - \frac{4y^5}{5} + y^4\right]_0^2 = \frac{144}{35}$$

19.



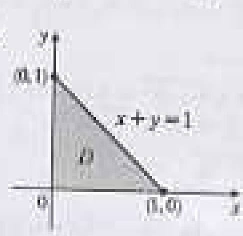
$$V = \int_1^2 \int_1^{7-3y} xy \, dx \, dy$$
$$= \int_1^2 \left[\tfrac{1}{2}x^2 y\right]_{x=1}^{x=7-3y} dy$$
$$= \tfrac{1}{2}\int_1^2 \left(48y - 42y^2 + 9y^3\right) dy$$
$$= \tfrac{1}{2}\left[24y^2 - 14y^3 + \tfrac{9}{4}y^4\right]_1^2 = \frac{31}{8}$$

20.



$$V = \int_0^1 \int_0^{1-x} \left(x^2 + y^2 + 4\right) dy \, dx = \int_0^1 \left[x^2 y + \tfrac{1}{3}y^3 + 4y\right]_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left[x^2 - x^3 + \tfrac{1}{3}(1-x)^3 + 4(1-x)\right] dx$$
$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}(1-x)^4 - 2(1-x^2)\right]_0^1 = \tfrac{13}{6}$$

21.



$$V = \int_0^1 \int_0^{1-x} (1 - x - y) \, dy \, dx$$
$$= \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left[(1-x)^2 - \tfrac{1}{2}(1-x)^2\right] dx$$
$$= \int_0^1 \tfrac{1}{2}(1-x)^2 \, dx = \left[-\tfrac{1}{6}(1-x)^3\right]_0^1 = \tfrac{1}{6}$$

The desired solid is shown in the first graph. From the second graph, we estimate that $y = \cos x$ intersects $y = x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$V \approx \int_0^{0.7391} \int_x^{\cos x} z\,dy\,dx = \int_0^{0.7391} \int_x^{\cos x} x\,dy\,dx = \int_0^{0.7391} [xy]_{y=x}^{y=\cos x}\,dx$$
$$= \int_0^{0.7391} \left(x\cos x - x^2\right) dx = \left[\cos x + x\sin x - \tfrac{1}{3}x^3\right]_0^{0.7391} \approx 0.1024$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y = 0$. In case you calculated the volume of this solid and want to check your work, its volume is $V \approx \int_0^{0.7391} \int_0^x x\,dy\,dx + \int_{0.7391}^{\pi/2} \int_0^{\cos x} x\,dy\,dx \approx 0.4684$.

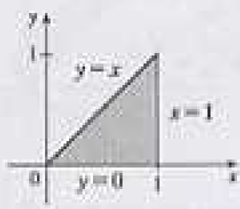
27. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume is

$$V = \int_0^2 \int_{x^3-x}^{x^2+x} z\,dy\,dx = \int_0^2 \int_{x^3-x}^{x^2+x} \left(x^3y^4 + xy^2\right) dy\,dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

28. For $|x| \le 1$ and $|y| \le 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. So the volume is given by

$$V = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\left(8 - x^2 - 2y^2\right) - \left(2x^2 + y^2\right)\right] dy\,dx = \frac{13\pi}{2} \quad \text{(using a CAS)}$$

29.



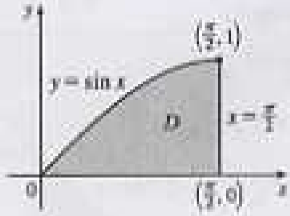
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le x, 0 \le x \le 1\}$$
$$= \{(x,y) \mid y \le x \le 1, 0 \le y \le 1\}$$

we have

$$\int_0^1 \int_0^x f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^1 \int_y^1 f(x,y)\,dx\,dy$$

30.



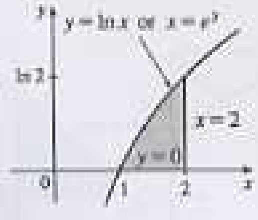
Because the region of integration is

$$D = \left\{(x,y) \mid 0 \le y \le \sin x, 0 \le x \le \tfrac{\pi}{2}\right\}$$
$$= \left\{(x,y) \mid \sin^{-1} y \le x \le \tfrac{\pi}{2}, 0 \le y \le 1\right\}$$

we have

$$\int_0^{\pi/2} \int_0^{\sin x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^1 \int_{\sin^{-1} y}^{\pi/2} f(x,y)\,dx\,dy$$

31.



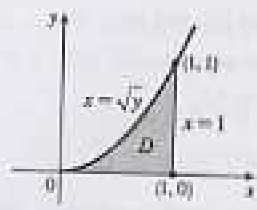
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \ln x, 1 \le x \le 2\}$$
$$= \{(x,y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$$

we have

$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^{\ln 2} \int_{e^y}^2 f(x,y)\,dx\,dy$$

36.

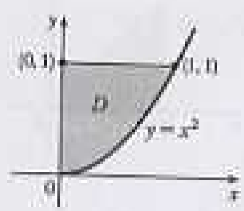


$$\int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy = \int_0^1\int_0^{x^2}\sqrt{x^3+1}\,dy\,dx$$
$$= \int_0^1\left[\sqrt{x^3+1}\,y\right]_{y=0}^{y=x^2}dx$$
$$= \int_0^1 x^2\sqrt{x^3+1}\,dx$$
$$= \tfrac{2}{9}\left(x^3+1\right)^{3/2}\Big]_0^1$$
$$= \tfrac{2}{9}\left(2^{3/2}-1\right)$$

37.

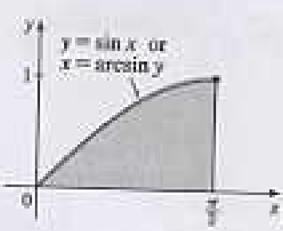
$$\int_0^3\int_{y^2}^9 y\cos x^2\,dx\,dy = \int_0^9\int_0^{\sqrt{x}} y\cos x^2\,dy\,dx$$
$$= \int_0^9 \cos x^2\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}}dx$$
$$= \int_0^9 \tfrac12 x\cos x^2\,dx = \tfrac14\sin x^2\Big]_0^9$$
$$= \tfrac14\sin 81$$

38.



$$\int_0^1\int_{x^2}^1 x^3\sin(y^3)\,dy\,dx = \int_0^1\int_0^{\sqrt{y}} x^3\sin(y^3)\,dx\,dy$$
$$= \int_0^1\left[\frac{x^4}{4}\sin(y^3)\right]_{x=0}^{x=\sqrt{y}}dy$$
$$= \int_0^1 \tfrac14 y^2\sin(y^3)\,dy$$
$$= -\tfrac{1}{12}\cos(y^3)\Big]_0^1 = \tfrac{1}{12}(1-\cos 1)$$

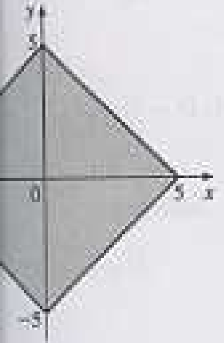
39.



$$\int_0^1\int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2}\int_0^{\sin x}\cos x\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2}\cos x\sqrt{1+\cos^2 x}\,\left[y\right]_{y=0}^{y=\sin x}dx$$
$$= \int_0^{\pi/2}\cos x\sqrt{1+\cos^2 x}\,\sin x\,dx$$
$$\left[\text{Let } u=\cos x,\ du=-\sin x\,dx,\ dx=du/(-\sin x)\right]$$
$$= \int_1^0 -u\sqrt{1+u^2}\,du = -\tfrac13\left(1+u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac13\left(\sqrt{8}-1\right) = \tfrac13\left(2\sqrt{2}-1\right)$$

x is an odd function of
an odd function of y and

$= 8\pi$



solid that lies below the
is equivalent to
ry of D. Thus, the solid is

$4x^2 = 0$, finding that

12.4 Double Integrals in Polar Coordinates

1. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 0 \le r \le 2, 0 \le \theta \le 2\pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 2-x\}$. Thus $\iint_R f(x,y)\,dA = \int_0^2\int_0^{2-x} f(x,y)\,dy\,dx$.

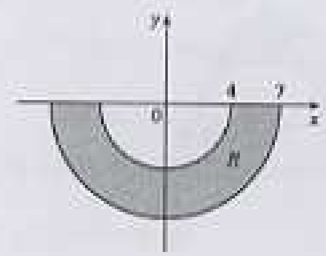
3. The region R is more easily described by rectangular coordinates: $R = \{(x,y) \mid -2 \le x \le 2, x \le y \le 2\}$. Thus $\iint_R f(x,y)\,dA = \int_{-2}^2\int_x^2 f(x,y)\,dy\,dx$.

4. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \frac{\pi}{2}\}$. Thus $\iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

5. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 5, 0 \le \theta \le 2\pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

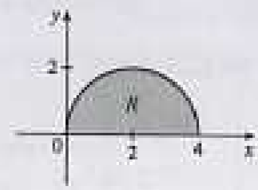
6. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \le r \le 2\sqrt{2}, \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\right\}$. Thus $\iint_R f(x,y)\,dA = \int_{\pi/4}^{3\pi/4}\int_0^{2\sqrt{2}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

7. The integral $\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 4 \le r \le 7, \pi \le \theta \le 2\pi\}$, the lower half of a ring.



$$\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\,dr\right) = [\theta]_\pi^{2\pi}\left[\frac{1}{2}r^2\right]_4^7 = \pi\cdot\frac{1}{2}(49-16) = \frac{33\pi}{2}$$

8. The integral $\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 0 \le r \le 4\cos\theta, 0 \le \theta \le \pi/2\}$. Since $r = 4\cos\theta \;\Leftrightarrow\; r^2 = 4r\cos\theta \;\Leftrightarrow\; x^2 + y^2 = 4x \;\Leftrightarrow\; (x-2)^2 + y^2 = 4$, R is the portion in the first quadrant of a circle of radius 2 with center $(2,0)$.



$$\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_{r=0}^{r=4\cos\theta} d\theta = \int_0^{\pi/2} 8\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} 4(1+\cos 2\theta)\,d\theta = 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2\pi$$

9. The disk D can be described in polar coordinates as $D = \{(r,\theta) \mid 0 \le r \le 3, 0 \le \theta \le 2\pi\}$. Then $\iint_D xy\,dA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^3 r^3\,dr\right) = \left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^3 = 0$.

10. $\iint_R \sqrt{x^2+y^2}\,dA = \int_0^\pi\int_1^3 \sqrt{r^2}\,r\,dr\,d\theta = \left(\int_0^\pi d\theta\right)\left(\int_1^3 r^2\,dr\right) = [\theta]_0^\pi\left[\frac{1}{3}r^3\right]_1^3 = \pi\left(\frac{27-1}{3}\right) = \frac{26}{3}\pi$

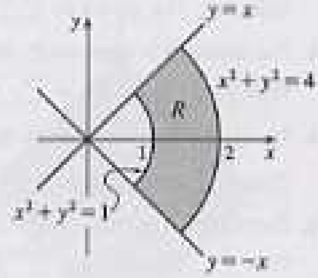
11. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\,dr\right)$

$$= [\theta]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)\left(e^{-4} - e^0\right) = \tfrac{\pi}{2}\left(1 - e^{-4}\right)$$

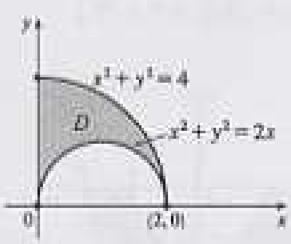
12. $\iint_R ye^x\,dA = \int_0^{\pi/2}\int_0^5 (r\sin\theta)e^{r\cos\theta}\,r\,dr\,d\theta = \int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr$. First we integrate $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta$: Let $u = r\cos\theta \Rightarrow du = -r\sin\theta\,d\theta$, and $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta = \int_{u=r}^{u=0} -re^u\,du = -r\left[e^0 - e^r\right] = re^r - r$. Then $\int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr = \int_0^5 (re^r - r)\,dr = \left[re^r - e^r - \frac{1}{2}r^2\right]_0^5 = 4e^5 - \frac{23}{2}$, where we integrated by parts in the first term.

13. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid -\pi/4 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus $\iint_R \arctan(y/x)\,dA = \int_{-\pi/4}^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$ since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, so the integral becomes

$$\int_{-\pi/4}^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \left(\int_{-\pi/4}^{\pi/4}\theta\,d\theta\right)\left(\int_1^2 r\,dr\right) = \left[\tfrac{1}{2}\theta^2\right]_{-\pi/4}^{\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2 = 0.$$



14.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\,y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \tfrac{8}{3} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$

$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0 + \tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

15. $V = \iint_{x^2+y^2\le 9}(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^3 (r^2)\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^3 r^3\,dr = [\theta]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^3 = 2\pi\left(\frac{81}{4}\right) = \frac{81\pi}{2}$

16. The sphere $x^2+y^2+z^2 = 16$ intersects the xy-plane in the circle $x^2+y^2 = 16$, so

$$V = 2\iint_{4\le x^2+y^2\le 16}\sqrt{16-x^2-y^2}\,dA \quad \text{(by symmetry)}$$

$$= 2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_2^4 r(16-r^2)^{1/2}\,dr$$

$$= 2[\theta]_0^{2\pi}\left[-\tfrac{1}{3}(16-r^2)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)\left(0 - 12^{3/2}\right) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

17. By symmetry,

$V = 2\iint_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr$

$= 2[\theta]_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0+\frac{1}{3}a^3\right) = \frac{4\pi}{3}a^3$

18. The paraboloid $z = 10 - 3x^2 - 3y^2$ intersects the plane $z = 4$ when $4 = 10 - 3x^2 - 3y^2$ or $x^2+y^2 = 2$. So

$$V = \iint_{x^2+y^2\le 2}\left[(10-3x^2-3y^2) - 4\right]dA = \int_0^{2\pi}\int_0^{\sqrt{2}}(6-3r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(6r-3r^3)\,dr = [\theta]_0^{2\pi}\left[3r^2 - \tfrac{3}{4}r^4\right]_0^{\sqrt{2}} = 6\pi$$

29. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x, y)$ to be the depth of the water at (x, y), then the volume of water in the pool is the volume of the solid that lies above $D = \{(x, y) \mid x^2 + y^2 \leq 400\}$ and below the graph of $f(x, y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0, -20) = 2$ to $f(0, 20) = 7$. The trace in the yz-plane is a line segment from $(0, -20, 2)$ to $(0, 20, 7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z - 7 = \frac{1}{8}(y - 20) \Rightarrow z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x, y)$ is independent of x, $f(x, y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x, y)\, dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r, \theta) \mid 0 \leq r \leq 20, 0 \leq \theta \leq 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta$$
$$= \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta = \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi}$$
$$= 1800\pi$$

Thus the pool contains $1800\pi \approx 5655\ \text{ft}^3$ of water.

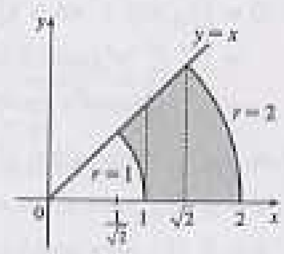
30. (a) The total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r} r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R re^{-r}\,dr = [\theta]_0^{2\pi}\left[-re^{-r} - e^{-r}\right]_0^R$$
$$= 2\pi\left[-Re^{-R} - e^{-R} + 0 + 1\right] = 2\pi\left(1 - Re^{-R} - e^{-R}\right)\ \text{ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1 - Re^{-R} - e^{-R}\right)}{R^2}\ \text{ft}^3$ (per hour per square foot). See the definition of the average value of a function on page 844.

31. $\int_{1/\sqrt{2}}^1\int_{\sqrt{1-x^2}}^x xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^2\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$

$$= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2} d\theta$$
$$= \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta\frac{15}{4} = \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



32. (a) $\iint_{D_a} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^a re^{-r^2}\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1 - e^{-a^2}\right)$ for each a. Then $\lim\limits_{a\to\infty}\pi\left(1 - e^{-a^2}\right) = \pi$ since $e^{-a^2} \to 0$ as $a \to \infty$. Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^a\int_{-a}^a e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right)$ for each a. Then, from (a), $\pi = \iint_{\mathbb{R}^2} - (x^2 + y^2)\,dA$, so

$\pi = \lim\limits_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA = \lim\limits_{a\to\infty}\left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

To evaluate $\lim\limits_{a\to\infty}\left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded. This is true since on $[-1, 1]$, $0 < e^{-x^2} \leq 1$ while on $(-\infty, -1)$, $0 < e^{-x^2} \leq e^x$ and on $(1, \infty)$, $0 < e^{-x^2} < e^{-x}$. Hence $0 \leq \int_{-\infty}^{\infty} e^{-x^2}\,dx \leq \int_{-\infty}^{-1} e^x\,dx + \int_{-1}^1 dx + \int_1^{\infty} e^{-x}\,dx = 2(e^{-1} + 1)$.