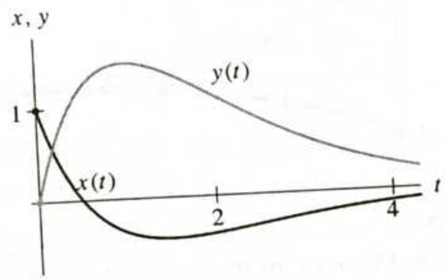
The solution corresponding to the initial condition $D = (1, 0)$ moves to the left and up through the first quadrant in the phase plane before it enters the second quadrant and heads toward the origin tangent to the line $y = -2x$. Thus the $y(t)$-graph is always positive for $t > 0$, and it attains a unique maximum value before it tends to 0. Initially the $x(t)$-graph decreases. It crosses the $y(t)$-graph, becomes negative, and attains a minimum value before it tends to 0 as $t \to \infty$.



EXERCISES FOR SECTION 3.6

1. The characteristic polynomial is

$$s^2 + 3s - 10,$$

so the eigenvalues are $s = 2$ and $s = -5$. Hence, the general solution is

$$y(t) = k_1 e^{2t} + k_2 e^{-5t}.$$

2. The characteristic polynomial is

$$s^2 + 6s + 8,$$

so the eigenvalues are $s = -2$ and $s = -4$. Hence, the general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-4t}.$$

3. The characteristic polynomial is

$$s^2 + 6s + 9,$$

so $s = -3$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{-3t} + k_2 t e^{-3t}.$$

From the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 11 \\ k_1 - 5k_2 = -7. \end{cases}$$

Solving for k_1 and k_2 yields $k_1 = 8$ and $k_2 = 3$. Hence, the solution to our initial-value problem is $y(t) = 8e^t + 3e^{-5t}$.

9. The characteristic polynomial is

$$s^2 + 2s + 5,$$

so the eigenvalues are $s = -1 \pm 2i$. Hence, the general solution is

$$y(t) = k_1 e^{-t} \cos 3t + k_2 e^{-t} \sin 3t.$$

From the initial condition $y(0) = 3$, we see that $k_1 = 3$. Differentiating

$$y(t) = 3e^{-t} \cos 3t + k_2 e^{-t} \sin 3t$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -3 + 2k_2$. Since $y'(0) = -1$, we have $k_2 = 1$. Hence, the solution to our initial-value problem is

$$y(t) = 3e^{-t} \cos 2t + e^{-t} \sin 2t.$$

10. The characteristic polynomial is

$$s^2 + 4s + 20,$$

so the eigenvalues are $s = -2 \pm 4i$. Hence, the general solution is

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t.$$

From the initial condition $y(0) = 2$, we see that $k_1 = 2$. Differentiating

$$y(t) = 2e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -4 + 4k_2$. Since $y'(0) = -8$, we have $k_2 = -1$. Hence the solution to our initial-value problem is

$$y(t) = 2e^{-2t} \cos 4t - e^{-2t} \sin 4t.$$

11. The characteristic polynomial is

$$s^2 + 2s + 1,$$

so $s = -1$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{-t} + k_2 t e^{-t}.$$

From the initial condition $y(0) = 1$, we see that $k_1 = 1$. Differentiating

$$y(t) = e^{-t} + k_2 t e^{-t}$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -1 + k_2$. Since $y'(0) = 1$, we have $k_2 = 2$. Hence, the solution to our initial-value problem is

$$y(t) = e^{-t} + 2te^{-t}.$$

12. The characteristic polynomial is

$$s^2 - 4s + 4,$$

so $s = 2$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{2t} + k_2 t e^{2t}.$$

From the initial condition $y(0) = 1$, we see that $k_1 = 1$. Differentiating $y(t) = e^{2t} + k_2 t e^{2t}$ and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = 2 + k_2$. Since $y'(0) = 1$, we have $k_2 = -1$. Hence, the solution to our initial-value problem is

$$y(t) = e^{2t} - te^{2t}.$$

13. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 7y = 0,$$

and the corresponding system is

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -7y - 8v. \end{aligned}$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 8\lambda + 7 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -7$.
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = -y \\ -7y - 8v = -v. \end{cases}$$

From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -7$ must satisfy the equation $v = -7y$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

(if $b > 2\sqrt{3}$) tends to 0 most quickly. This determination depends upon the value of b for which $-b/2$ (if $b < 2\sqrt{3}$) or $(-b + \sqrt{b^2 - 12})/2$ (if $b > 2\sqrt{3}$) is most negative.

For $0 < b < 2\sqrt{3}$, $-b/2$ is decreasing. Using calculus, we can show that $(-b + \sqrt{b^2 - 12})/2$ is increasing for $b > 2\sqrt{3}$. Therefore, we must examine the rate if $b = 2\sqrt{3}$. In this case, we have repeated eigenvalues, and the typical solution is a linear combination of terms of the form $e^{-\sqrt{3}t}$ and $te^{-\sqrt{3}t}$. Again using calculus, we can check that both of these solutions tend to 0 faster than $e^{-\alpha t}$ where $\alpha \neq 2\sqrt{3}$.

35. The characteristic equation for this harmonic oscillator is

$$s^2 + bs + 3 = 0,$$

and the roots are

$$s_1 = \frac{-b - \sqrt{b^2 - 12}}{2} \quad \text{and} \quad s_2 = \frac{-b + \sqrt{b^2 - 12}}{2}.$$

If $b^2 < 12$, these roots are complex. In this case, all solutions include a factor of the form $e^{(-b/2)t}$, and they tend to the equilibrium at this rate.

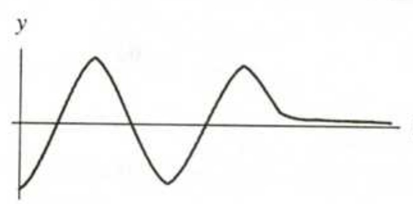
If $b^2 > 12$, the roots are real, and the general solution is

$$y(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}.$$

The first exponential in this expression tends to 0 most quickly, so if $k_2 = 0$, we have solutions that tend to 0 at the rate of $e^{s_1 t}$. This rate is the quickest approach to 0.

The roots are repeated if $b^2 - 12 = 0$, that is, if $b = 2\sqrt{3}$. The fastest approach is then given by a term of the form $e^{-\sqrt{3}t}$.

36. **(a)**



(b) Using the model of a harmonic oscillator for the suspension system, the corresponding system has either real or complex eigenvalues. If it has complex eigenvalues, then solutions spiral in the phase plane and oscillations of $y(t)$ continue for all time. If there are real eigenvalues, then solutions do not spiral, and in fact, they cannot cross the v-axis (where $y = 0$) more than once. Hence, the behavior described is impossible for a harmonic oscillator.

(c) There is room for disagreement in this answer. One reasonable choice is an oscillator with complex eigenvalues and some damping so that the system does oscillate, but the amplitude of the oscillations decays sufficiently rapidly so that only the first two "bounces" are of significant size.

37. **(a)** Since the fluid causes the object to accelerate as it moves and the force causing this acceleration is proportional to the velocity, the force equation for this "mass-spring" system is

$$m\frac{d^2y}{dt^2} = -ky + b_{mf}\frac{dy}{dt},$$

which can be written as

$$m\frac{d^2y}{dt^2} - b_{mf}\frac{dy}{dt} + ky = 0.$$

(b) The equivalent first-order system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -\frac{k}{m}y + \frac{b_{mf}}{m}v.$$

(c) The characteristic equation is

$$m\lambda^2 - b_{mf}\lambda + k = 0,$$

and the eigenvalues are

$$\frac{b_{mf} \pm \sqrt{b_{mf}^2 - 4mk}}{2m}.$$

Since m, b_{mf}, and k are all positive parameters, the eigenvalues are either positive real numbers or complex numbers with a positive real part. If both eigenvalues are real, then the origin is called an "overstimulated" source. The magnitudes of $y(t)$ and $v(t)$ tend to infinity without oscillation. If the eigenvalues are complex, then the origin is a spiral source and the oscillator is called understimulated. The solutions spiral away from the origin with natural period $4m\pi/\sqrt{b_{mf}^2 - 4mk}$.

38. We have the second-order differential equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

The characteristic polynomial is

$$m\lambda^2 + b\lambda + k,$$

and the eigenvalues are

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

In our case, $b^2 - 4mk < 0$, so the eigenvalues can be written as

$$\frac{-b \pm i\sqrt{4mk - b^2}}{2m}$$

Using this expression for the eigenvalues, we obtain the natural period P as

$$P = \frac{2\pi}{\dfrac{\sqrt{4mk - b^2}}{2m}} = \frac{4\pi}{\sqrt{4mk - b^2}}.$$

(a) If $m = 1$, $k = 2$, and $b = 1$, we have a natural period of $4\pi/\sqrt{7}$.

EXERCISES FOR SECTION 4.1

1. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 6s + 8,$$

so the eigenvalues are $s = -2$ and $s = -4$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} + k_2 e^{-4t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2 y_p}{dt^2} + 6\frac{dy_p}{dt} + 8y_p &= ke^{-t} - 6ke^{-t} + 8ke^{-t} \\ &= 3ke^{-t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/3$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-4t} + \tfrac{1}{3} e^{-t}.$$

2. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 6s + 8,$$

so the eigenvalues are $s = -2$ and $s = -4$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} + k_2 e^{-4t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-3t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2 y_p}{dt^2} + 6\frac{dy_p}{dt} + 8y_p &= 9ke^{-3t} - 18ke^{-3t} + 8ke^{-3t} \\ &= -ke^{-3t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = -2$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-4t} - 2e^{-3t}.$$

3. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 7s + 12,$$

so the eigenvalues are $s = -3$ and $s = -4$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-3t} + k_2 e^{-4t}.$$

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To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-2t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 7\frac{dy_p}{dt} + 12y_p &= 4ke^{-2t} - 14ke^{-2t} + 12ke^{-2t}\\ &= 2ke^{-2t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 3/2$. The general solution of the forced equation is

$$y(t) = k_1e^{-3t} + k_2e^{-4t} + \tfrac{3}{2}e^{-2t}.$$

4. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 4s + 13,$$

so the eigenvalues are $s = -2 \pm 3i$. Hence, the general solution of the homogeneous equation is

$$k_1e^{-2t}\cos 3t + k_2e^{-2t}\sin 3t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 4\frac{dy_p}{dt} + 13y_p &= ke^{-t} - 4ke^{-t} + 13ke^{-t}\\ &= 10ke^{-t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/10$. The general solution of the forced equation is

$$y(t) = k_1e^{-2t}\cos 3t + k_2e^{-2t}\sin 3t + \tfrac{1}{10}e^{-t}.$$

5. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 4s + 13,$$

so the eigenvalues are $s = -2 \pm 3i$. Hence, the general solution of the homogeneous equation is

$$k_1e^{-2t}\cos 3t + k_2e^{-2t}\sin 3t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-2t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 4\frac{dy_p}{dt} + 13y_p &= 4ke^{-2t} - 8ke^{-2t} + 13ke^{-2t}\\ &= 9ke^{-2t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = -1/3$. The general solution of the forced equation is

$$y(t) = k_1e^{-2t}\cos 3t + k_2e^{-2t}\sin 3t - \tfrac{1}{3}e^{-2t}.$$

Solving, we have $k_1 = 15/2$ and $k_2 = -6$, so the solution of the initial-value problem is

$$y(t) = \tfrac{15}{2}e^{-3t} - 6e^{-4t} + \tfrac{1}{2}e^{-t}.$$

11. This is the same equation as Exercise 5. The general solution is

$$y(t) = k_1 e^{-2t} \cos 3t + k_2 e^{-2t} \sin 3t - \tfrac{1}{3}e^{-2t}.$$

To find the solution with the initial conditions $y(0) = y'(0) = 0$, we compute

$$y'(t) = -2k_1 e^{-2t} \cos 3t - 3k_1 e^{-2t} \sin 3t - 2k_2 e^{-2t} \sin 3t + 3k_2 e^{-2t} \cos 3t + \tfrac{2}{3}e^{-2t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 - \frac{1}{3} = 0 \\ -2k_1 + 3k_2 + \frac{2}{3} = 0. \end{cases}$$

Solving, we have $k_1 = 1/3$ and $k_2 = 0$, so the solution of the initial-value problem is

$$y(t) = \tfrac{1}{3}e^{-2t} \cos 3t - \tfrac{1}{3}e^{-2t}.$$

12. This is the same equation as Exercise 6. The general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-5t} + \tfrac{1}{3}te^{-2t}.$$

To find the solution with the initial conditions $y(0) = y'(0) = 0$, we compute

$$y'(t) = -2k_1 e^{-2t} - 5k_2 e^{-5t} + \tfrac{1}{3}e^{-2t} - \tfrac{2}{3}te^{-2t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 0 \\ -2k_1 - 5k_2 + \frac{1}{3} = 0. \end{cases}$$

Solving, we have $k_1 = -1/9$ and $k_2 = 1/9$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{1}{9}e^{-2t} + \tfrac{1}{9}e^{-5t} + \tfrac{1}{3}te^{-2t}.$$

13. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 3.$$

So the eigenvalues are $s = -1$ and $s = -3$, and the general solution of the unforced equation is

$$k_1 e^{-t} + k_2 e^{-3t}.$$

(b) The derivative of the general solution is

$$y'(t) = -k_1 e^{-t} - 3k_2 e^{-3t} + 2e^{-2t}.$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 - 1 = 0 \\ -k_1 - 3k_2 + 2 = 0. \end{cases}$$

Solving, we find that $k_1 = 1/2$ and $k_2 = 1/2$, so the solution of the initial-value problem is

$$y(t) = \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-3t} - e^{-2t}.$$

(c) In the general solution, all three terms tend to zero, so the solution tends to zero. We can say a little more by noting that the term $k_1 e^{-t}$ is much larger (provided $k_1 \neq 0$). Hence, most solutions tend to zero at the rate of e^{-t}. If $k_1 = 0$, then solutions tend to zero at the rate of e^{-3t} provided $k_2 \neq 0$.

15. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 3.$$

So the eigenvalues are $s = -1$ and $s = -3$, and the general solution of the unforced equation is

$$k_1 e^{-t} + k_2 e^{-3t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-4t}$. Substituting $y_p(t)$ into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 3y_p &= 16ke^{-4t} - 16ke^{-4t} + 3ke^{-4t} \\ &= 3ke^{-4t}. \end{aligned}$$

So $k = 1/3$ yields a solution of the forced equation.
The general solution of the forced equation is therefore

$$y(t) = k_1 e^{-t} + k_2 e^{-3t} + \tfrac{1}{3}e^{-4t}.$$

(b) The derivative of the general solution is

$$y'(t) = -k_1 e^{-t} - 3k_2 e^{-3t} - \tfrac{4}{3}e^{-4t}.$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{1}{3} = 0 \\ -k_1 - 3k_2 - \frac{4}{3} = 0. \end{cases}$$

Solving, we find that $k_1 = 1/6$ and $k_2 = -1/2$, so the solution of the initial-value problem is

$$y(t) = \tfrac{1}{6}e^{-t} - \tfrac{1}{2}e^{-3t} + \tfrac{1}{3}e^{-4t}.$$

(c) In the general solution, all three terms tend to zero, so the solution tends to zero. We can say a little more by noting that the term $k_1 e^{-t}$ is much larger (provided $k_1 \neq 0$). Hence, most solutions tend to zero at the rate of e^{-t}. If $k_1 = 0$, then solutions tend to zero at the rate of e^{-3t} provided $k_2 \neq 0$.

Solving, we find that $k_1 = -1/20$ and $k_2 = 1/40$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{1}{20}e^{-2t}\cos 4t + \tfrac{1}{40}e^{-2t}\sin 4t + \tfrac{1}{20}e^{-4t}.$$

(c) From the formula for the general solution, we see that every solution tends to zero. The e^{-4t} term in the general solution tends to zero quickest, so for large t, the solution is very close to the unforced solution. All solutions tend to zero and all but the purely exponential one oscillates with frequency $2/\pi$ and an amplitude that decreases at the rate of e^{-2t}.

19. The natural guesses of $y_p(t) = ke^{-t}$ and $y_p(t) = kte^{-t}$ fail to be solutions of the forced equation because they are both solutions of the unforced equation. (The characteristic polynomial of the unforced equation is

$$s^2 + 2s + 1,$$

which has -1 as a double root.)

So we guess $y_p(t) = kt^2e^{-t}$. Substituting this guess into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 2\frac{dy_p}{dt} + y_p &= (2ke^{-t} - 4kte^{-t} + kt^2e^{-t}) + 2(2kte^{-t} - kt^2e^{-t}) + kt^2e^{-t} \\ &= 2ke^{-t}.\end{aligned}$$

So $k = 1/2$ yields the solution

$$y_p(t) = \tfrac{1}{2}t^2e^{-t}.$$

From the characteristic polynomial, we know that the general solution of the unforced equation is

$$k_1e^{-t} + k_2te^{-t}.$$

Consequently, the general solution of the forced equation is

$$y(t) = k_1e^{-t} + k_2te^{-t} + \tfrac{1}{2}t^2e^{-t}.$$

20. If we guess a constant function of the form $y_p(t) = k$, then substituting $y_p(t)$ into the left-hand side of the differential equation yields

$$\begin{aligned}\frac{d^2(k)}{dt^2} + p\frac{d(k)}{dt} + qk &= 0 + 0 + qk \\ &= qk.\end{aligned}$$

Since the right-hand side of the differential equation is simply the constant c, $k = c/q$ yields a constant solution.

21. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 6s + 8.$$

So the eigenvalues are $s = -2$ and $s = -4$, and the general solution of the unforced equation is

$$k_1e^{-2t} + k_2e^{-4t}.$$

To find one solution of the forced equation, we guess the constant function $y_p(t) = k$. Substituting $y_p(t)$ into the left-hand side of the differential equation, we obtain

$$\frac{d^2y_p}{dt^2} + 6\frac{dy_p}{dt} + 8y_p = 0 + 6 \cdot 0 + 8k = 8k.$$

Hence, $k = 5/8$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1e^{-2t} + k_2e^{-4t} + \tfrac{5}{8}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = -2k_1e^{-2t} - 4k_2e^{-4t}.$$

Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{5}{8} = 0 \\ -2k_1 - 4k_2 = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -5/4$ and $k_2 = 5/8$. The solution of the initial-value problem is

$$y(t) = -\tfrac{5}{4}e^{-2t} + \tfrac{5}{8}e^{-4t} + \tfrac{5}{8}.$$

22. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 5s + 6.$$

So the eigenvalues are $s = -2$ and $s = -3$, and the general solution of the unforced equation is

$$k_1e^{-2t} + k_2e^{-3t}.$$

To find one solution of the forced equation, we guess the constant function $y_p(t) = k$. Substituting $y_p(t)$ into the left-hand side of the differential equation, we obtain

$$\frac{d^2y_p}{dt^2} + 5\frac{dy_p}{dt} + 6y_p = 0 + 5 \cdot 0 + 6k = 6k.$$

Hence, $k = 1/3$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1e^{-2t} + k_2e^{-3t} + \tfrac{1}{3}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = -2k_1e^{-2t} - 3k_2e^{-3t}.$$

Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{1}{3} = 0 \\ -2k_1 - 3k_2 = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -1$ and $k_2 = 2/3$. The solution of the initial-value problem is

$$y(t) = -e^{-2t} + \tfrac{2}{3}e^{-3t} + \tfrac{1}{3}.$$

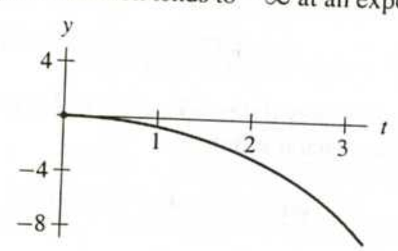
Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 - \frac{1}{3} = 0 \\ \sqrt{2}\,k_2 - \frac{1}{3} = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = 1/3$ and $k_2 = \sqrt{2}/6$. The solution of the initial-value problem is

$$y(t) = \tfrac{1}{3}\cos\sqrt{2}\,t + \tfrac{\sqrt{2}}{6}\sin\sqrt{2}\,t - \tfrac{1}{3}e^t.$$

(c) Since $e^t \to \infty$ quickly, the solution tends to $-\infty$ at an exponential rate.



31. **(a)** The general solution for the homogeneous equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

Suppose $y_p(t) = at^2 + bt + c$. Substituting $y_p(t)$ into the differential equation, we get

$$\frac{d^2y_p}{dt^2} + 4y_p = -3t^2 + 2t + 3$$

$$2a + 4(at^2 + bt + c) = -3t^2 + 2t + 3$$

$$4at^2 + 4bt + (2a + 4c) = -3t^2 + 2t + 3.$$

Therefore, $y_p(t)$ is a solution if and only if

$$\begin{cases} 4a = -3 \\ 4b = 2 \\ 2a + 4c = 3. \end{cases}$$

Therefore, $a = -3/4$, $b = 1/2$, and $c = 9/8$. The general solution is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t - \tfrac{3}{4}t^2 + \tfrac{1}{2}t + \tfrac{9}{8}.$$

(b) To solve the initial-value problem, we use the initial conditions $y(0) = 2$ and $y'(0) = 0$ along with the general solution to form the simultaneous equations

$$\begin{cases} k_1 + \frac{9}{8} = 2 \\ 2k_2 + \frac{1}{2} = 0. \end{cases}$$

Therefore, $k_1 = 7/8$ and $k_2 = -1/4$. The solution is

$$y(t) = \tfrac{7}{8}\cos 2t - \tfrac{1}{4}\sin 2t - \tfrac{3}{4}t^2 + \tfrac{1}{2}t + \tfrac{9}{8}.$$

by multiplying the particular solution we found in Exercise 6 by $-1/2$. Hence the general solution of the differential equation in this exercise is

$$y(t) = k_1 e^{-4t} + k_2 e^{-2t} - \tfrac{2}{325}\cos 3t + \tfrac{36}{325}\sin 3t.$$

To obtain the desired initial conditions, we must solve for k_1 and k_2. We have

$$\begin{cases} k_1 + k_2 - \tfrac{2}{325} = 0 \\ -4k_1 - 2k_2 + \tfrac{108}{324} = 0. \end{cases}$$

We obtain $k_1 = 4/25$ and $k_2 = -2/13$. The desired solution is

$$y(t) = \tfrac{4}{25}e^{-4t} - \tfrac{2}{13}e^{-2t} - \tfrac{2}{325}\cos 3t + \tfrac{36}{325}\sin 3t.$$

13. From Exercise 9, we know that the general solution of this equation is

$$y(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t - \tfrac{3}{20}\sin 2t + \tfrac{3}{40}\cos 2t.$$

To find the desired solution, we must solve for k_1 and k_2 using the initial conditions. We have

$$\begin{cases} k_1 + \tfrac{3}{40} = 0 \\ -2k_1 + 4k_2 - \tfrac{6}{20} = 0. \end{cases}$$

We obtain $k_1 = -3/40$ and $k_2 = 3/80$. The desired solution is

$$y(t) = -\tfrac{3}{40}e^{-2t}\cos 4t + \tfrac{3}{80}e^{-2t}\sin 4t - \tfrac{3}{20}\sin 2t + \tfrac{3}{40}\cos 2t.$$

14. First we find the general solution of the differential equation using the Extended Linearity Principle and the standard guess-and-test technique. The complex version of the equation is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2e^{2it},$$

and we guess $y_c(t) = ae^{2it}$ as a particular solution. Substituting this guess into the equation yields

$$a = \frac{2}{-3+4i} = \frac{-6-8i}{25}.$$

Hence, a particular solution is the real part of

$$y_c(t) = \frac{-6-8i}{25}(\cos 2t + i\sin 2t).$$

We have

$$y(t) = -\tfrac{6}{25}\cos 2t + \tfrac{8}{25}\sin 2t.$$

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is $s^2 + 2s + 1$, which has $s = -1$ as a double root. Hence, the general solution of the original equation is

$$y(t) = k_1 e^{-t} + k_2 t e^{-t} - \tfrac{6}{25}\cos 2t + \tfrac{8}{25}\sin 2t.$$

To obtain the desired initial conditions, we solve for k_1 and k_2 using

$$\begin{cases} k_1 - \frac{6}{25} = 0 \\ -k_1 + k_2 + \frac{16}{25} = 0. \end{cases}$$

We see that $k_1 = 6/25$ and $k_2 = -2/5$, so the desired solution is

$$y(t) = \tfrac{6}{25}e^{-t} - \tfrac{2}{5}te^{-t} - \tfrac{6}{25}\cos 2t + \tfrac{8}{25}\sin 2t.$$

15. (a) If we guess

$$y_p(t) = a\cos 3t + b\sin 3t,$$

then

$$y_p'(t) = -3a\sin 3t + 3b\cos 3t$$

and

$$y_p''(t) = -9a\cos 3t - 9b\sin 3t.$$

Substituting this guess and its derivatives into the differential equation gives

$$(-8a + 9b)\cos 3t + (-9a - 8b)\sin 3t = \cos 3t.$$

Thus $y_p(t)$ is a solution if a and b satisfy the simultaneous equations

$$\begin{cases} -8a + 9b = 1 \\ -9a - 8b = 0. \end{cases}$$

Solving these equations for a and b, we obtain $a = -8/145$ and $b = 9/145$, so

$$y_p(t) = -\tfrac{8}{145}\cos 3t + \tfrac{9}{145}\sin 3t$$

is a solution.

(b) If we guess

$$y_p(t) = A\cos(3t + \phi),$$

then

$$y_p'(t) = -3A\sin(3t + \phi)$$

and

$$y_p''(t) = -9A\cos(3t + \phi).$$

Substituting this guess and its derivatives into the differential equation yields

$$-8A\cos(3t + \phi) - 9A\sin(3t + \phi) = \cos 3t.$$

Using the trigonometric identities for the sine and cosine of the sum of two angles, we have

$$-8A\,(\cos 3t\cos\phi - \sin 3t\sin\phi) - 9A\,(\sin 3t\cos\phi + \cos 3t\sin\phi) = \cos 3t.$$

This equation can be rewritten as

$$(-8A\cos\phi - 9A\sin\phi)\cos 3t + (8A\sin\phi - 9A\cos\phi)\sin 3t = \cos 3t.$$

It holds if

$$\begin{cases} -8A\cos\phi - 9A\sin\phi = 1 \\ 9A\cos\phi - 8A\sin\phi = 0. \end{cases}$$

Multiplying the first equation by 9 and the second by 8 and adding yields

$$145A\sin\phi = -9.$$

Similarly, multiplying the first equation by -8 and the second by 9 and adding yields

$$145A\cos\phi = -8.$$

Taking the ratio gives

$$\frac{\sin\phi}{\cos\phi} = \tan\phi = \frac{9}{8}.$$

Also, squaring both $145A\sin\phi = -9$ and $145A\cos\phi = -8$ yields

$$145^2A^2\cos^2\phi + 145^2A^2\sin^2\phi = 145,$$

so $A^2 = 1/145$.
We can use either $A = 1/\sqrt{145}$ or $A = -1/\sqrt{145}$, but this choice of sign for A effects the value of ϕ. If we pick $A = -1/\sqrt{145}$, then $\sqrt{145}\,\sin\phi = 9$, $\sqrt{145}\,\cos\phi = 8$, and $\tan\phi = 9/8$. In this case, $\phi = \arctan(9/8)$. Hence, a particular solution of the original equation is

$$y_p(t) = \frac{1}{\sqrt{145}}\cos\left(3t + \arctan\frac{9}{8}\right).$$

16. Due to the damping ($p > 0$), the solution of the homogeneous equation (the unforced response) tends to zero. Solutions of the nonhomogeneous equation (the forced response) are sums of terms involving $\cos t$ and $\sin t$. Hence they oscillate with period 2π. This observation eliminates Figures (i) and (iv).

Figures (ii) and (iii) differ in the amplitude of the oscillations of the forced response. We compute a particular solution of the equation as usual, considering the complex version

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = e^{it}.$$

We obtain a particular solution of the form $y_c(t) = ae^{it}$ by substituting $y_c(t)$ into the differential equation. We have

$$(2+5i)ae^{it} = e^{it},$$

which is satisfied if $a = 1/(2+5i)$. Since

$$y_c(t) = \frac{2-5i}{29}\,(\cos t + i\sin t),$$

a particular solution of the forced equation is

$$y(t) = \tfrac{2}{29}\cos t + \tfrac{5}{29}\sin t.$$

Due to the damping, all solutions have the same long-term behavior including the long-term amplitude of the oscillations. We have

$$|y(t)| = |a| = 1/\sqrt{29} \approx 0.19.$$

This observation eliminates the graphs shown in Figure (ii), and our analysis is consistent with the graphs shown in Figure (iii).

17. Due to the damping ($p > 0$), the solution of the homogeneous equation (the unforced response) tends to zero. Solutions of the nonhomogeneous equation (the forced response) are sums of terms involving $\cos t$ and $\sin t$. Hence they oscillate with period 2π. This observation eliminates Figures (i) and (iv).

Figures (ii) and (iii) differ in the amplitude of the oscillations of the forced response. We compute a particular solution of the equation as usual, considering the complex version

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = e^{it}.$$

We obtain a particular solution of the form $y_c(t) = ae^{it}$ by substituting $y_c(t)$ into the differential equation. We have

$$(2+i)ae^{it} = e^{it},$$

which is satisfied if $a = 1/(2+i)$. Since

$$y_c(t) = \frac{2-i}{5}\,(\cos t + i\sin t),$$

a particular solution of the forced equation is

$$y(t) = \tfrac{2}{5}\cos t + \tfrac{1}{5}\sin t.$$

Due to the damping, all solutions have the same long-term behavior including the long-term amplitude of the oscillations. Since the amplitude of $y(t)$ is $|a| = 1/\sqrt{5} \approx 0.44$, the graphs shown in Figure (iii) are not possible. Our analysis is consistent with the graphs shown in Figure (ii).

18. Due to the damping ($p > 0$), the solution of the homogeneous equation (the unforced response) tends to zero. Solutions of the nonhomogeneous equation (the forced response) are sums of terms involving $\cos 3t$ and $\sin 3t$. Hence they oscillate with period $2\pi/3$. This observation eliminates Figures (ii) and (iii).

The difference between the graphs in Figure (i) and the graphs in Figure (iv) is the rate at which solutions tend toward each other. This rate is the same as the rate at which the solutions of the homogeneous equation tend to zero. The characteristic polynomial of the homogeneous equation is $s^2 + 5s + 1$, so the eigenvalues are $s = (-5 \pm \sqrt{21})/2 \approx (-5 \pm 4.6)/2$. The rate that most solutions tend to zero is determined by the larger of the two negative eigenvalues. In this case, most solutions tend to zero at the rate of $e^{-0.2t}$. This rate is slow, which is consistent with Figure (i).

19. Due to the damping ($p > 0$), the solution of the homogeneous equation (the unforced response) tends to zero. Solutions of the nonhomogeneous equation (the forced response) are sums of terms involving $\cos 3t$ and $\sin 3t$. Hence they oscillate with period $2\pi/3$. This observation eliminates Figures (ii) and (iii).

The difference between the graphs in Figure (i) and the graphs in Figure (iv) is the rate at which solutions tend toward each other. This rate is the same as the rate at which the solutions of the homogeneous equation tend to zero. The characteristic polynomial of the homogeneous equation is $s^2 + s + 1$, so the eigenvalues are $s = (-1 \pm \sqrt{3}\,i)/2$. The rate that solutions tend to zero is determined by the real part of these complex eigenvalues. Hence, all solutions of the homogeneous equation tend to zero at the rate of $e^{-0.5t}$, which is definitely faster than the rate in Exercise 18. The graphs in Figure (iv) are consistent with this analysis.

20. Substituting $ky_p(t)$ into the left-hand side of the differential equation and simplifying yields

$$\frac{d^2(ky_p)}{dt^2} + p\frac{d(ky_p)}{dt} + q(ky_p) = k\frac{d^2y_p}{dt^2} + pk\frac{dy_p}{dt} + qky_p$$

since the derivative of ky is $k(dy/dt)$ if k is a constant. Consequently,

$$\begin{aligned}\frac{d^2(ky_p)}{dt^2} + p\frac{d(ky_p)}{dt} + q(ky_p) &= k\left(\frac{d^2y_p}{dt^2} + p\frac{dy_p}{dt} + qy_p\right)\\ &= k\,g(t)\end{aligned}$$

because $y_p(t)$ is a solution of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t).$$

21. By Exercise 5 we know that one solution of

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = \cos t$$

is

$$y_1(t) = \tfrac{7}{85}\cos t + \tfrac{6}{85}\sin t.$$

Using the result of Exercise 20, a particular solution of the given equation is $y_2(t) = 5y_1(t)$. In other words,

$$y_2(t) = \tfrac{7}{17}\cos t + \tfrac{6}{17}\sin t$$

is a particular solution to the equation in this exercise.

The general solution of the homogeneous equation is the same as in Exercise 5, so the general solution for this exercise is

$$y(t) = k_1e^{-4t} + k_2e^{-2t} + \tfrac{7}{17}\cos t + \tfrac{6}{17}\sin t.$$

22. **(a)** Due to the result in Exercise 36 in Section 4.1, we consider each forcing term separately. Th is, we consider the two differential equations

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 3 \quad \text{and} \quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 2\cos 2t.$$

To find a particular solution of the first equation, we guess a constant function $y_1(t) = a$. Sub stituting this guess into the equation yields $20a = 3$, so $y_1(t) = 3/20$ is a solution.

To find a particular solution of the second equation, we consider the complex version

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 2e^{2it}$$

and guess a solution of the form $y_c(t) = ae^{2it}$. Substituting $y_c(t)$ into the equation yields

$$(16 + 8i)ae^{2it} = 2e^{2it},$$

which is satisfied if $a = 2/(16 + 8i)$. A solution $y_2(t)$ of the second equation is obtained by taking the real part of $y_c(t)$. Since

$$y_c(t) = \left(\tfrac{1}{10} - \tfrac{1}{20}i\right)\left(\cos 2t + i\sin 2t\right),$$

we have

$$y_2(t) = \tfrac{1}{10}\cos 2t + \tfrac{1}{20}\sin 2t.$$

To find the solution of the unforced equation, we note that the characteristic polynomial is $s^2 + 4s + 20$, which has roots $s = -2 \pm 4i$.

Hence, the general solution is

$$y(t) = k_1e^{-2t}\cos 4t + k_2e^{-2t}\sin 4t + \tfrac{3}{20} + \tfrac{1}{10}\cos 2t + \tfrac{1}{20}\sin 2t.$$

(b) The first two terms of the general solution tend quickly to zero, so all solutions eventually oscillate about $y = 3/20$. The period and amplitude of the oscillations is determined by the period and amplitude of the oscillations of $y_2(t)$. The period of $y_2(t)$ is π.

23. **(a)** Using the fact that the real part of $e^{(-1+i)t}$ is $e^{-t}\cos t$, the complex version of this equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{(-1+i)t}.$$

Guessing $y_c(t) = ae^{(-1+i)t}$ yields

$$a(-1+i)^2e^{(-1+i)t} + 4a(-1+i)e^{(-1+i)t} + 20ae^{(-1+i)t} = e^{(-1+i)t}.$$

Simplifying we have

$$a(16 + 2i)e^{(-1+i)t} = e^{(-1+i)t}.$$

Thus, $y_c(t)$ is a solution of the complex differential equation if $a = 1/(16 + 2i)$, and we have

$$y_c(t) = \left(\tfrac{4}{65} - \tfrac{1}{130}i\right)e^{-t}\left(\cos t + i\sin t\right).$$

So one solution of the original equation is

$$y_p(t) = \tfrac{4}{65}e^{-t}\cos t + \tfrac{1}{130}e^{-t}\sin t.$$

To find the general solution of the homogeneous equation, we note that the characteristic polynomial $s^2 + 4s + 20$ has roots $s = -2 \pm 4i$.
Hence, the general solution of the original equation is

$$y(t) = k_1 e^{-2t}\cos 4t + k_1 e^{-2t}\sin 4t + \tfrac{4}{65}e^{-t}\cos t + \tfrac{1}{130}e^{-t}\sin t.$$

(b) All four terms in the general solution tend to zero as $t \to \infty$. Hence, all solutions tend to zero as $t \to \infty$. The terms with factors of e^{-2t} tend to zero very quickly, which leaves the terms of the particular solution $y_p(t)$ as the largest terms, so all solutions are asymptotic to $y_p(t)$. Since the solution $y_p(t)$ oscillates with period 2π and the amplitude of its oscillations decreases at the rate of e^{-t}, all solutions oscillate with this period and decaying amplitude.

24. **(a)** To find the general solution of the unforced equation, we note that the characteristic polynomial is $s^2 + 4s + 20$, which has roots $s = -2 \pm 4i$.
To find a particular solution of the forced equation, we note that the complex version of the equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{(-2+4i)t}.$$

We could guess $y_c(t) = ae^{(-2+4i)t}$ as a particular solution, but with perfect hindsight, we recall that the roots of the characteristic polynomial of the unforced equation are $-2 \pm 4i$. Hence, $e^{(-2+4i)t}$ is already a solution of the homogeneous equation. In other words, no value of a will make this $y_c(t)$ a solution. (Why?)
So we second guess $y_c(t) = ate^{(-2+4i)t}$ and substitute this guess into the equation to obtain

$$a\left([-4 + 8i + (-12 - 16i)t] + 4[1 + (-2 + 4i)t] + 20t\right)e^{(-2+4i)t} = e^{(-2+4i)t},$$

which simplifies to

$$a(8i)e^{(-2+4i)t} = e^{(-2+4i)t}.$$

Hence, $y_c(t) = ate^{(-2+4i)t}$ is a solution if $a = 1/(8i) = -i/8$. To find a particular solution of the original equation, we compute the imaginary part of

$$y_c(t) = -\tfrac{1}{8}ite^{-2t}(\cos 4t + i\sin 4t).$$

We have

$$y(t) = -\tfrac{1}{8}te^{-2t}\cos 4t.$$

The general solution of the original equation is

$$y(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t - \tfrac{1}{8}te^{-2t}\cos 4t.$$

(b) All terms of the general solution tend to zero. The term that tends to zero most slowly is $-(te^{-2t}\cos 4t)/8$, so for large t, all solutions are approximately equal to this term.