.I -n

EM2b

E&M Ch 8 - Conservation Laws

p.345

Thus 12 Apr 07
week 2

8.1.1 CHARGE CONSERVATION:

$$\frac{dQ}{dt} = \frac{\partial}{\partial t}\int \rho \, d\tau = I_{out} = -\oint J \cdot da = -\int \rho v \cdot da$$

$$\int \frac{\partial \rho}{\partial t} d\tau = -\int \nabla \cdot J \, d\tau \quad \text{(divergence thm)}$$

$$\underline{\frac{\partial \rho}{\partial t} = -\nabla \cdot \bar{J}} = -\nabla \cdot (\rho \bar{v})$$

Chen; convective derivative: $\frac{dG}{dt} = \frac{\partial G}{\partial t} + (u \cdot \nabla) G$

change in G in a frame moving with particle

change in G at a point in fixed space

change in G as the observer moves into a region where G varies

8.1.2 POYNTING'S THEOREM

Electromagnetic work done $= \frac{dW}{dt} = -\frac{dU_{em}}{dt} - \bar{\nabla} \cdot \bar{S}$

stored

(leaking)

where stored $U_{em} = \int u_{em} \, d\tau$

energy density of fields $u_{em} = \frac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0)$

Poynting vector $=$ $\frac{\text{Power flow}}{\text{area}} = \bar{S} = \frac{1}{\mu_0} E \times B$

$$\underline{\frac{\partial (u_{mech} + u_{em})}{\partial t} = -\bar{\nabla} \cdot \bar{S}}$$

Convective derivative - Chen - Plasma Physics

velocity **v**. The fluid equation is obtained simply by multiplying Eq. [3-29] by the density n:

$$mn\frac{d\mathbf{u}}{dt} = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad [3\text{-}30]$$

This is, however, not a convenient form to use. In Eq. [3-29], the time derivative is to be taken *at the position of the particles*. On the other hand, we wish to have an equation for fluid elements *fixed in space*, because it would be impractical to do otherwise. Consider a drop of cream in a cup of coffee as a fluid element. As the coffee is stirred, the drop distorts into a filament and finally disperses all over the cup, losing its identity. A fluid element at a fixed spot in the cup, however, retains its identity although particles continually go in and out of it.

To make the transformation to variables in a fixed frame, consider $\mathbf{G}(x, t)$ to be any property of a fluid in one-dimensional x space. The change of $\mathbf{G}$ with time *in a frame moving with the fluid* is the sum of two terms:

$$\frac{d\mathbf{G}(x,t)}{dt} = \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \mathbf{G}}{\partial x}\frac{dx}{dt} = \frac{\partial \mathbf{G}}{\partial t} + u_x\frac{\partial \mathbf{G}}{\partial x} \qquad [3\text{-}31]$$

The first term on the right represents the change of **G** at a fixed point in space, and the second term represents the change of **G** as the observer moves with the fluid into a region in which **G** is different. In three dimensions, Eq. [3-31] generalizes to

$$\frac{d\mathbf{G}}{dt} = \frac{\partial \mathbf{G}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{G} \qquad [3\text{-}32]$$

This is called the *convective derivative* and is sometimes written $D\mathbf{G}/Dt$. Note that $(\mathbf{u} \cdot \nabla)$ is a *scalar* differential operator. Since the sign of this term is sometimes a source of confusion, we give two simple examples.

Figure 3-1 shows an electric water heater in which the hot water has risen to the top and the cold water has sunk to the bottom. Let $G(x, t)$ be the temperature T; ∇G is then upward. Consider a fluid element near the edge of the tank. If the heater element is turned on, the fluid element is heated as it moves, and we have $dT/dt > 0$. If, in addition, a paddle wheel sets up a flow pattern as shown, the temperature in a *fixed* fluid element is lowered by the convection of cold water from the bottom. In this case, we have $\partial T/\partial x > 0$ and $u_x > 0$, so that $\mathbf{u} \cdot \nabla T > 0$. The temperature change in the fixed element, $\partial T/\partial t$, is given by a balance

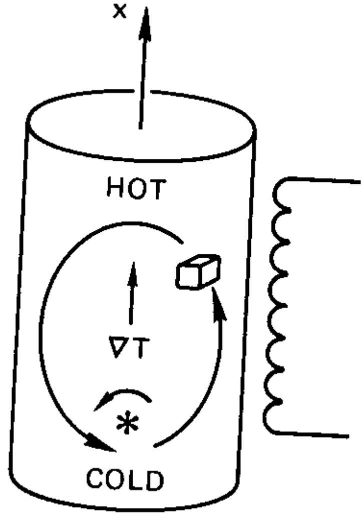


FIGURE 3-1 **Motion of fluid elements in a hot water heater.**

of these effects,

$$\frac{\partial T}{\partial t}=\frac{dT}{dt}-\mathbf{u}\cdot\nabla T \tag{3-33}$$

It is quite clear that $\partial T/\partial t$ can be made zero, at least for a short time.

As a second example we may take G to be the salinity S of the water near the mouth of a river (Fig. 3-2). If x is the upstream direction, there

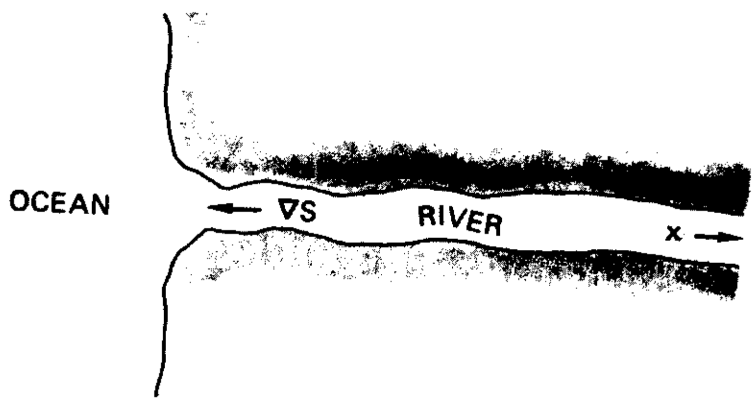


FIGURE 3-2 **Direction of the salinity gradient at the mouth of a river.**

is normally a gradient of S such that $\partial S/\partial x < 0$. When the tide comes in, the entire interface between salt and fresh water moves upstream, and $u_x > 0$. Thus

$$\frac{\partial S}{\partial t} = -u_x \frac{\partial S}{\partial x} > 0 \qquad [3\text{-}34]$$

meaning that the salinity increases at any given point. Of course, if it rains, the salinity decreases everywhere, and a negative term dS/dt is to be added to the middle part of Eq. [3-34].

As a final example, take G to be the density of cars near a freeway entrance at rush hour. A driver will see the density around him increasing as he approaches the crowded freeway. This is the convective term $(\mathbf{u} \cdot \nabla)G$. At the same time, the local streets may be filling with cars that enter from driveways, so that the density will increase even if the observer does not move. This is the $\partial G/\partial t$ term. The total increase seen by the observer is the sum of these effects.

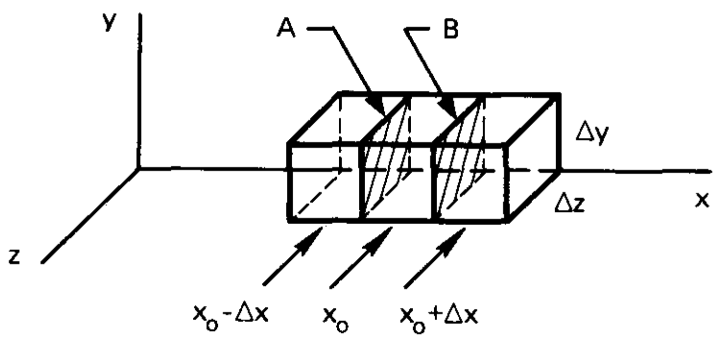
In the case of a plasma, we take $\mathbf{G}$ to be the fluid velocity $\mathbf{u}$ and write Eq. [3-30] as

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad [3\text{-}35]$$

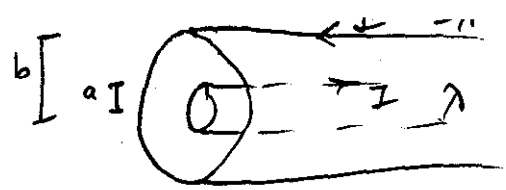
where $\partial \mathbf{u}/\partial t$ is the time derivative in a fixed frame.

The Stress Tensor 3.3.2

When thermal motions are taken into account, a pressure force has to be added to the right-hand side of Eq. [3-35]. This force arises from the



Origin of the elements of the stress tensor. **FIGURE 3-3**



8.1 (a) $\frac{\text{Power}}{\text{Area}} = S = \frac{|\mathbb{E} \times B|}{\mu_0}$ So Power $= \int S \cdot da$

$$P = \int_a^b S \cdot 2\pi s \, ds$$

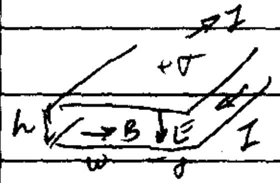
$$S = \frac{1}{\mu_0} \frac{\lambda}{2\pi\epsilon_0 s} \hat{s} \times \frac{\mu_0 I}{2\pi s} \hat{\phi} = \qquad \hat{z}$$

$P =$

$$V = \int_a^b E \cdot dl =$$

So $P = IV$

(b)



We found $\mathbb{E} = \frac{\sigma}{\epsilon_0} = \frac{V}{h}$ in direction drawn

and $B = \frac{\mu_0 I}{w}$ in direction drawn

Power $= \int S \cdot da = Swh$ where $S = \frac{1}{\mu_0} |\mathbb{E} \times \mathbb{B}| =$

$P =$

$V = \int \mathbb{E} \cdot dl =$

So $P = IV$