Problem 1.37 Express the unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.64). Check your answers several ways ($\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \stackrel{?}{=} 1$, $\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} \stackrel{?}{=} 0$, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \stackrel{?}{=} \hat{\boldsymbol{\phi}}, \ldots$). Also work out the inverse formulas, giving $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ (and θ, ϕ).

$$\left.\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \end{aligned}\right\} \tag{1.64}$$

Problem 1.38

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

b) Do the same for $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob. 1.16.)

Problem 1.41 Express the cylindrical unit vectors $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.75). "Invert" your formulas to get $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{s}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ (and ϕ).

$$\left.\begin{aligned} \hat{\mathbf{s}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}. \end{aligned}\right\} \tag{1.75}$$

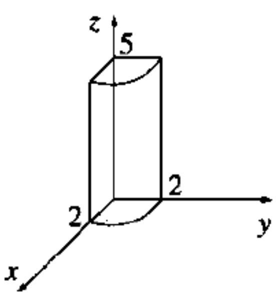
Problem 1.42

(a) Find the divergence of the function

$$\mathbf{v} = s(2 + \sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of $\mathbf{v}$.

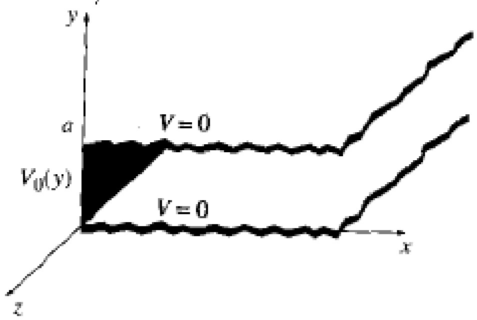


$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$$

$$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Problem 3.12 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0, and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$.



$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0(y) \text{ when } x = 0, \\ \text{(iv)} & V \to 0 \text{ as } x \to \infty. \end{array}\right\}$$

Problem 3.13 For the infinite slot (Ex. 3.3) determine the charge density $\sigma(y)$ on the strip at $x = 0$, assuming it is a conductor at constant potential V_0.

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a). \tag{3.36}$$

Problem 3.23 Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find *all* solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

Problem 3.24 Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field $\mathbf{E}_0$. Find the surface charge induced on the pipe. [Use your result from Prob. 3.23.]