Because charge is conserved, whatever flows out through the surface must come at the expense of that remaining inside:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau.$$

(The minus sign reflects the fact that an *outward* flow *decreases* the charge left in $\mathcal{V}$.) Since this applies to *any* volume, we conclude that

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.} \tag{5.29}$$

This is the precise mathematical statement of local charge conservation; it is called the **continuity equation**.

For future reference, let me summarize the "dictionary" we have implicitly developed for translating equations into the forms appropriate to point, line, surface, and volume currents:

$$\sum_{i=1}^{n} (\;\;) q_i \mathbf{v}_i \sim \int_{\text{line}} (\;\;) \mathbf{I}\, dl \sim \int_{\text{surface}} (\;\;) \mathbf{K}\, da \sim \int_{\text{volume}} (\;\;) \mathbf{J}\, d\tau. \tag{5.30}$$

This correspondence, which is analogous to $q \sim \lambda\, dl \sim \sigma\, da \sim \rho\, d\tau$ for the various charge distributions, generates Eqs. 5.15, 5.24, and 5.27 from the original Lorentz force law (5.1).

Problem 5.4 Suppose that the magnetic field in some region has the form

$$\mathbf{B} = kz\,\hat{\mathbf{x}}$$

(where k is a constant). Find the force on a square loop (side a), lying in the yz plane and centered at the origin, if it carries a current I, flowing counterclockwise, when you look down the x axis.

Problem 5.5 A current I flows down a wire of radius a.

(a) If it is uniformly distributed over the surface, what is the surface current density K?

(b) If it is distributed in such a way that the volume current density is inversely proportional to the distance from the axis, what is J?

Problem 5.6

(a) A phonograph record carries a uniform density of "static electricity" σ. If it rotates at angular velocity ω, what is the surface current density K at a distance r from the center?

(b) A uniformly charged solid sphere, of radius R and total charge Q, is centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\mathbf{J}$ at any point (r, θ, ϕ) within the sphere.

Problem 5.7 For a configuration of charges and currents confined within a volume $\mathcal{V}$, show that

$$\int_{\mathcal{V}} \mathbf{J}\, d\tau = d\mathbf{p}/dt,$$

where $\mathbf{p}$ is the total dipole moment. [*Hint:* evaluate $\int_{\mathcal{V}} \nabla \cdot (x\mathbf{J})\, d\tau$.]

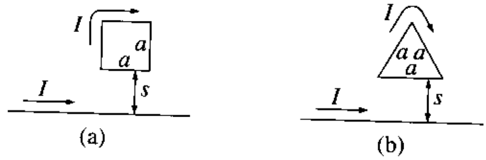

Figure 5.24

Problem 5.10

(a) Find the force on a square loop placed as shown in Fig. 5.24(a), near an infinite straight wire. Both the loop and the wire carry a steady current I.

(b) Find the force on the triangular loop in Fig. 5.24(b).

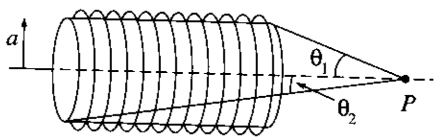

Figure 5.25

Problem 5.11 Find the magnetic field at point P on the axis of a tightly wound solenoid (helical coil) consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I (Fig. 5.25). Express your answer in terms of θ_1 and θ_2 (it's easiest that way). Consider the turns to be essentially circular, and use the result of Ex. 5.6. What is the field on the axis of an *infinite* solenoid (infinite in both directions)?

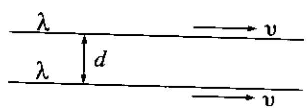

Figure 5.26

Problem 5.12 Suppose you have two infinite straight line charges λ, a distance d apart, moving along at a constant speed v (Fig. 5.26). How great would v have to be in order for the magnetic attraction to balance the electrical repulsion? Work out the actual number... Is this a reasonable sort of speed?[8]

---

[8] If you've studied special relativity, you may be tempted to look for complexities in this problem that are not really there—λ and v are both measured *in the laboratory frame*, and this is *ordinary electrostatics* (see footnote 4).

Problem 5.13 A steady current I flows down a long cylindrical wire of radius a (Fig. 5.40). Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that J is proportional to s, the distance from the axis.

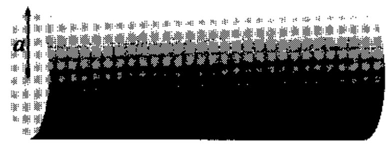

Figure 5.40

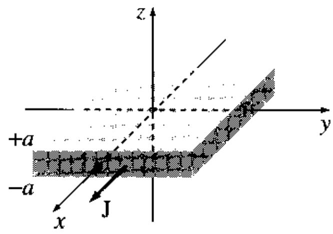

Figure 5.41

Problem 5.14 A thick slab extending from $z = -a$ to $z = +a$ carries a uniform volume current $\mathbf{J} = J\,\hat{\mathbf{x}}$ (Fig. 5.41). Find the magnetic field, as a function of z, both inside and outside the slab.

Problem 5.15 Two long coaxial solenoids each carry current I, but in opposite directions, as shown in Fig. 5.42. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find $\mathbf{B}$ in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

Figure 5.42

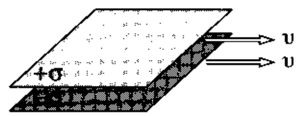

Figure 5.43

Problem 5.16 A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Fig. 5.43.

(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed v would the magnetic force balance the electrical force?[11]

[11] See footnote 8.

! **Problem 5.17** Show that the magnetic field of an infinite solenoid runs parallel to the axis, *regardless of the cross-sectional shape of the coil*, as long as that shape is constant along the length of the solenoid. What is the magnitude of the field, inside and outside of such a coil? Show that the toroid field (5.58) reduces to the solenoid field, when the radius of the donut is so large that a segment can be considered essentially straight.

Problem 5.18 In calculating the current enclosed by an amperian loop, one must, in general, evaluate an integral of the form

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{a}.$$

The trouble is, there are infinitely many surfaces that share the same boundary line. Which one are we supposed to use?

5.3.4 Comparison of Magnetostatics and Electrostatics

The divergence and curl of the *electrostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = 0, & \text{(no name).} \end{cases}$$

These are **Maxwell's equations** for electrostatics. Together with the boundary condition $\mathbf{E} \to 0$ far from all charges, Maxwell's equations determine the field, if the source charge density ρ is given; they contain essentially the same information as Coulomb's law plus the principle of superposition. The divergence and curl of the *magnetostatic* field are

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

These are Maxwell's equations for magnetostatics. Again, together with the boundary condition $\mathbf{B} \to 0$ far from all currents, Maxwell's equations determine the magnetic field; they are equivalent to the Biot-Savart law (plus superposition). Maxwell's equations and the force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

constitute the most elegant formulation of electrostatics and magnetostatics.

The electric field *diverges away from* a (positive) charge; the magnetic field line *curls around* a current (Fig. 5.44). Electric field lines originate on positive charges and terminate on negative ones; magnetic field lines do not begin or end anywhere—to do so would require a nonzero divergence. They either form closed loops or extend out to infinity. To put it another way, *there are no point sources for* **B**, as there are for **E**; there exists no magnetic analog to electric charge. This is the physical content of the statement $\nabla \cdot \mathbf{B} = 0$. Coulomb and others believed that magnetism was produced by **magnetic charges** (**magnetic monopoles**, as we would now call them), and in some older books you will still