Problem 1.38 $\int_{volume} (\bar\nabla\cdot\bar V)\,d\tau = \oint_{area} \bar V\cdot d\bar a$

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

b) Do the same for $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob. 1.16.)

Spherical coordinates -

$$\bar\nabla\cdot\bar V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}$$

(a) $$\bar\nabla\cdot\bar V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) = \frac{1}{r^2}(4r^3) = 4r$$

$\int(\bar\nabla\cdot\bar V)\,d\tau = \int 4r\,d\tau$ where $d\tau = r^2\,dr\,\sin\theta\,d\theta\,d\Phi$

$$\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\Phi = -2\pi\cos\theta\Big|_0^\pi = -2\pi(-1-1) = 4\pi$$

$$= \int_0^R 4r\,4\pi r^2\,dr$$
$$= 16\pi\int_0^R r^3\,dr = 16\pi\,\frac{r^4}{4}\Big|_0^R$$

$$\int(\bar\nabla\cdot\bar V)\,d\tau = 4\pi R^4.$$

Now check $\oint\bar V\cdot d\bar a$ where $da = r^2\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\Phi = 4\pi r^2$

$$\oint\bar V\cdot d\bar a = r^2\,4\pi r^2\Big|_{r=R} = 4\pi R^4 \checkmark$$

(b) $$\bar\nabla\cdot\bar V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\left(\frac{\partial}{\partial r}1\right) = 0, \text{ so } \int(\bar\nabla\cdot\bar V)\,d\tau = 0$$

$$\oint\bar V\cdot d\bar a = \frac{1}{r^2}\,4\pi r^2 = 4\pi \quad ? \quad \text{These don't agree?!}$$

$$\nabla\cdot\left(\frac{\hat r}{r^2}\right) = 0 \text{ for } r\neq 0, \quad = \infty \text{ at } r = 0$$

$\int\left(\bar\nabla\cdot\frac{\hat r}{r^2}\right)d\tau = 4\pi$ is correct

Problem 1.42

(a) Find the divergence of the function

$$\mathbf{v} = s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}.$$

$$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\frac{\partial V_z}{\partial z} = 3$$

$$\frac{\partial V_\varphi}{\partial \varphi} = s\left[\sin\varphi(-\sin\varphi) + \cos\varphi(\cos\varphi)\right] = s\left[\cos^2\varphi - \sin^2\varphi\right]$$

$$\frac{\partial}{\partial s}(sV_s) = \frac{\partial}{\partial s} s^2(2+\sin^2\varphi) = 2s(2+\sin^2\varphi)$$

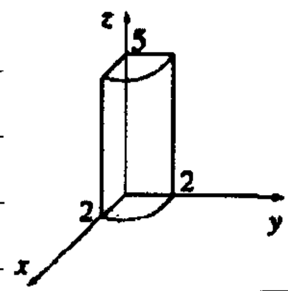
$$\nabla\cdot V = 2(2+\sin^2\varphi) + [\cos^2\varphi - \sin^2\varphi] + 3$$

$$= 4 + 2\sin^2\varphi - \sin^2\varphi + \cos^2\varphi + 3$$

$$= 4 + \sin^2\varphi + \cos^2\varphi + 3$$

$$= 4 + 1 + 3 = 8$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.



$$\int_{\mathcal{V}} (\nabla\cdot\mathbf{v})\,d\tau = \oint_{\mathcal{S}} \mathbf{v}\cdot d\mathbf{a}$$

$$\int d\tau = \int_0^2 s\,ds \int_0^{\pi/2} d\varphi \int_0^5 dz$$

$$\int(\nabla\cdot V)\,d\tau = \int_0^2 8\,s\,ds \int_0^{\pi/2} d\varphi \int_0^5 dz$$

$$= 8\cdot 2\cdot\frac{\pi}{2}\cdot 5 = 8\pi\cdot 5 = 40\pi$$

$$d\bar{a}_{top} = s\,ds\,d\varphi\,\hat{z},\quad z=5$$

$$\bar{V}\cdot d\bar{a} = 3z\,s\,ds\,d\varphi = 15\,s\,ds\,d\varphi$$

$$\int \bar{V}\cdot d\bar{a} = 15\int_0^2 s\,ds\int_0^{\pi/2} d\varphi = 15\cdot 2\cdot\frac{\pi}{2} = \underline{15\pi}$$

TOP

$da_{bottom} = -s\, ds\, d\varphi\, \hat{z}, \quad z=0$

$\bar{v}\cdot d\bar{a} = -3z s\, ds\, d\varphi = 0 \rightarrow \int \bar{v}\cdot da = 0$ / bottom

$da_{front} = s\, d\varphi\, dz\, \hat{s}, \quad s = 2$

$\bar{v}\cdot d\bar{a} = v_s \cdot da = s(z+\sin^2\varphi)\, s\, d\varphi\, dz = 4(z+\sin^2\varphi)\, d\varphi\, dz$

$\int \bar{v}\cdot d\bar{a} = 4\int_0^{\pi/2}(z+\sin^2\varphi)\, d\varphi \int_0^5 dz = 20\left(\pi+\frac{\pi}{4}\right) = 20\cdot\frac{5}{4}\pi$

$= 25\pi$ / front

$da_{back} = ds\, dz\, \hat{\varphi}, \quad \varphi = \frac{\pi}{2}$

$\bar{v}\cdot d\bar{a} = v_\varphi \cdot da = s\sin\varphi\cos\varphi\, ds\, dz = 0 \rightarrow \int v\cdot da = 0$ / back

$da_{left\ side} = -ds\, dz\, \hat{\varphi}, \quad \varphi = 0$

$\bar{v}\cdot d\bar{a} = v_\varphi \cdot d\bar{a} = -s\sin\varphi\cos\varphi\, ds\, dz = 0 \rightarrow \int v\cdot da = 0$ / side

$\int \bar{v}\cdot d\bar{a} = 15\pi + 25\pi = 40\pi = \int (\nabla\cdot\bar{v})\, d\tau$ ✓

Satisfies divergence thm

(c) Find the curl of v.

$$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

EM HW #4 – Physical Systems due 20 Feb 2007
3.12, 13, 23, 24

Problem 3.12 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0, and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$. See attached worksheet

Problem 3.13 For the infinite slot (Ex. 3.3) determine the charge density $\sigma(y)$ on the strip at $x = 0$, assuming it is a conductor at constant potential V_0.

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5...} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a). \tag{3.36}$$

Recall from Ch. 2 that the E field changes across a charge distribution: $E = \frac{\sigma}{\epsilon_0} = \frac{-\partial V}{\partial n} = \frac{-\partial V}{\partial x}$ in this case

$$\frac{\partial V}{\partial x} = \frac{4V_0}{\pi} \sum \frac{1}{n} \underbrace{\frac{\partial}{\partial x} e^{-n\pi x/a}}_{-\frac{n\pi}{a} e^{-n\pi x/a}} \sin\frac{n\pi y}{a} \Bigg|_{x=0}$$

$$\frac{\partial V}{\partial x} = -\frac{4V_0}{a} \sum \underbrace{e^{-n\pi x/a}}_{\to 1 @ x=0} \sin\frac{n\pi y}{a} \Bigg|_{x=0}$$

$$\frac{\partial V}{\partial x} = -\frac{4V_0}{a} \sum \sin\frac{n\pi y}{a}$$

so $\sigma = -\epsilon_0 \frac{\partial V}{\partial x} = +\frac{4V_0\epsilon_0}{a} \sum_{n=1,3,5} \sin\frac{n\pi y}{a}$

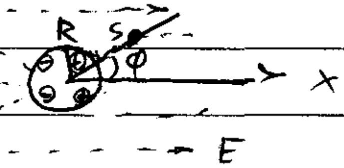
Problem 3.23 Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find *all* solutions to the radial equation; in particular, your result must accommodate the case of an infinite line charge, for which (of course) we already know the answer.]

See attached worksheet. Solution:

$$V = \underbrace{C \ln s + D}_{k=0} + \sum_{k=1}^{\infty} \left(A s^{k} + B s^{-k}\right) e^{\pm ik\phi}$$

Problem 3.24 Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field $\mathbf{E}_0$. Find the surface charge induced on the pipe. [Use your result from Prob. 3.23.]

This is much like Ex 3.8, p. 141



The conducting pipe is an equipotential

Far from the pipe, $V \to -E_0 x + C$

(i) $V(s \gg R) = -E_0 s \cos\phi$

(ii) We can set $V(s=R) = 0$

We must fit these boundary conditions to the solution of the laplacian in cylindrical coordinates, above:

$$V = C \ln s + D + \sum_{k=1}^{\infty} \left(A s^{k} + B s^{-k}\right)\left(c \cos k\phi - d \sin k\phi\right)$$

C and D = 0 because $V = 0$ at $s = R$ - no constant terms

$d = 0$ because of the orientation of the E field - no sines.

(i) And the only wave number is $k = 1$, since $V \sim \cos\phi$

so $V = \left(a s + \frac{b}{s}\right)\cos\phi$ where I combined $a = \frac{A}{c}$, $b = Bc$

BC (ii) $V(s=R) = 0 = \left(aR + \frac{b}{R}\right)\cos\phi \rightarrow aR^2 = -b$

BC (i) $V(s \gg R) = \left(as + \frac{b}{\infty}\right)\cos\phi \simeq as\cos\phi = -E_0 s \cos\phi$

$a = -E_0 \rightarrow b = +E_0 R^2$

$$V = \left(as + \frac{b}{s}\right)\cos\phi = \left(-E_0 s + \frac{E_0 R^2}{s}\right)\cos\phi$$

$$V = E_0 s\left(-1 + \frac{R^2}{s^2}\right)\cos\phi$$

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial s}\Big|_{s=R} = -\epsilon_0 E_0 \cos\phi \frac{\partial}{\partial s}\left[s\left(-1 + \frac{R^2}{s^2}\right)\right]$$

$$\frac{\partial}{\partial s}\left[-s + \frac{R^2}{s}\right] = -1 - \frac{R^2}{s^2}$$

$$\sigma = +\epsilon_0 E_0 \cos\phi\left(1 + \frac{R^2}{s^2}\right)\Big|_{s=R} = \epsilon_0 E_0 \cos\phi(1+1)$$

$$\sigma = 2\epsilon_0 E_0 \cos\phi$$