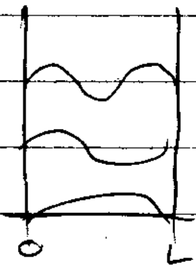
Modern Physics HW #5 due 1 Mar 2007 E/Z

(1) Expectation values in square well (from lecture) (2) Griffiths QM 2.13
(3) 2.14

(1)

Expectation values



Exercise: Consider the infinite square well of width L.

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

(a) What is <x>? $= \int_{-\infty}^{\infty} x\,\psi^2\,dx$ A: L/2

(b) What is <x²>? $= \int_{-\infty}^{\infty} x^2\,\psi^2\,dx$ $B: \langle x^2\rangle = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$

(c) What is <p>? (Guess first) C: <p>=0 $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = -i\hbar\int_{-\infty}^{\infty}\left(\psi^*\frac{\partial\psi}{\partial x}\right)dx$

(d) What is <p²>? (Guess first) D: <p²>=2mE

(a) By symmetry, $\langle x\rangle = \frac{L}{2}$. Calculating it,

$\langle x\rangle = \frac{2}{L}\int_0^L x\,\sin^2\frac{n\pi x}{L}\,dx$ Let $y = \frac{n\pi x}{L}$, $dy = \frac{n\pi}{L}dx$ (Limit $x=L$: $y = n\pi$

$x = \frac{Ly}{n\pi}$ $\frac{L}{n\pi}dy = dx$

$\langle x\rangle = \frac{2}{L}\int\left(\frac{Ly}{n\pi}\right)\sin^2 y\left(\frac{L}{n\pi}\right)dy$

$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\int y\sin^2 y\,dy$. Schaum (14.348) p. 76

$\int_0^{n\pi} y\sin^2 y\,dy = \frac{y^2}{4} - \frac{y\sin 2y}{4} - \frac{\cos 2y}{8}\Big|_0^{n\pi}$

$= \frac{1}{4}(n\pi)^2 - \frac{1}{4}(n\pi\sin 2n\pi) - \frac{1}{8}(\cos 2n\pi - \cos 0)$

$= \frac{1}{4}(n\pi)^2 - 0 - 0$

$\langle x\rangle = \frac{2L}{(n\pi)^2}\,\frac{1}{4}(n\pi)^2 = \frac{L}{2}$ ✓

as predicted by symmetry.

*Problem 2.13 Using the methods and results of this section, QHO

(a) Normalize ψ_1 (Equation 2.51) by direct integration. Check your answer a[...] the general formula (Equation 2.54). *Note*: In this and most problems inv[...] the harmonic oscillator, it simplifies the notation if you introduce the va[...] $\xi \equiv \sqrt{m\omega/\hbar}\,x$ and the constant $\alpha \equiv (m\omega/\pi\hbar)^{1/4}$.

(b) Find ψ_2, but don't bother to normalize it.

(c) Sketch ψ_0, ψ_1, and ψ_2.

(d) Check the orthogonality of ψ_0, ψ_1, and ψ_2. *Note*: If you exploit the eve[...] and oddness of the functions, there is really only one integral left to ev[...] explicitly.

$$\psi_0(x) = A_0 e^{-m\omega x^2/2\hbar}$$

$$\psi_1(x) = A_1\sqrt{\frac{m\omega}{\hbar}}\,x e^{-m\omega x^2/2\hbar}$$

$$\psi_2(x) = A_2\left(1 - \frac{2m\omega x^2}{\hbar}\right)e^{-m\omega x^2/2\hbar}$$

(a) $\int_{-\infty}^{\infty} \psi_1^2\,dx = 1 = A_1^2 \frac{m\omega}{\hbar}\int x^2 e^{-ax^2}\,dx$ where $a = m\omega/2\hbar$

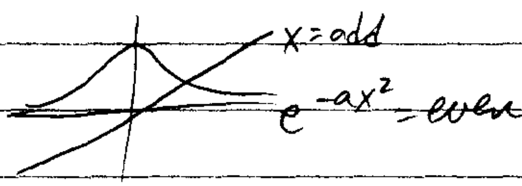
DWIGHT $\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx =$

$1 = A_1^2 \frac{m\omega}{\hbar}$

$A_1 =$

(d) $\int \psi_0\psi_1\,dx = A_0A_1\sqrt{\frac{m\omega}{\hbar}}\int x e^{-2m\omega x^2/2\hbar}\,dx = 0$ ✓

because



$\int \psi_0\psi_2 = A_0A_2\int e^{-ax^2}(1 - 2b^2x^2)e^{-ax^2}\,dx$ where $b = \sqrt{\frac{m\omega}{\hbar}}$

$= A_0A_2\left[\int e^{-2ax^2}\,dx - 2b^2\int x^2 e^{-2ax^2}\,dx\right]$

Both these integrands are even, so we have to evaluate them.

DWIGHT $\int e^{-ax^2}\,dx =$

$\int e^{-2ax^2}\,dx =$

DWIGHT $\int x^2 e^{-ax^2} dx =$

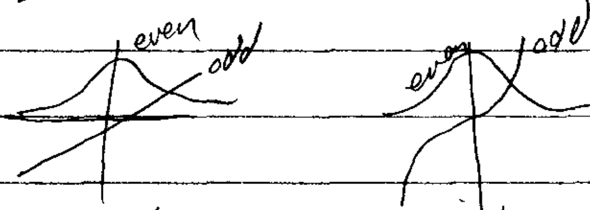
$\int x^2 e^{-2ax^2} dx =$

$\int \psi_0 \psi_2 =$

$$\int_{-\infty}^{\infty} \psi_1 \psi_2 \, dx = A_1 A_2 b \int_{-\infty}^{\infty} x(1-2b^2x^2) e^{-2ax^2} dx$$

$$= A_1 A_2 b \left[\int x e^{-2ax^2} dx - 2b^2 \int x^3 e^{-2ax^2} dx \right] = 0$$

because



integrates ≠ 0 integrates to 0

So ψ_1 and ψ_2 are orthogonal

***Problem 2.14** Using the results of Problem 2.13, QHO

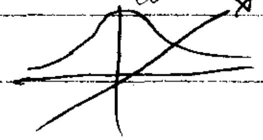
$\langle x \rangle = 0$ by symmetry

(a) Compute $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, and $\langle p^2 \rangle$, for the states ψ_0 and ψ_1.

(b) Check the uncertainty principle for these states.

(c) Compute $\langle T \rangle$ and $\langle V \rangle$ for these states (no new integration allowed!). Is their sum what you would expect?

(a) $\int x \psi_0^2 \, dx = A_0^2 \int_{-\infty}^{\infty} x e^{-2ax^2} \, dx$ where $a = \frac{-m\omega}{2\hbar}$, $A_0 =$



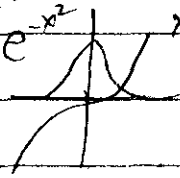
therefore $\langle x \rangle = 0$ ✓

ψ_0: $\langle p \rangle = m \frac{d\langle x \rangle}{dt} = 0$

ψ_0: $\langle x^2 \rangle = A_0^2 \int_{-\infty}^{\infty} x^2 e^{-ax^2} \, dx =$

$\langle x^2 \rangle =$

ψ_1 $\int x \psi_1^2 \, dx = A_1^2 \frac{m\omega}{\hbar} \int_{-\infty}^{\infty} x \, x^2 e^{-2ax^2} \, dx = 0$ because



then $\langle p_1 \rangle = 0 = m \, d\langle x_1 \rangle / dt$

$\langle x^2 \rangle = A_1^2 \frac{m\omega}{\hbar} \int x^4 e^{-2ax^2} \, dx =$

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ψ_0: $\langle p^2 \rangle \neq 2mE$ because $V \neq 0$ here.

$$p = -i\hbar \frac{\partial}{\partial x}, \quad p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$\frac{\partial}{\partial x} \psi_0 = \frac{\partial}{\partial x} A_0 e^{-ax^2} = -2a A_0 e^{-ax^2}$$

$$\frac{\partial^2}{\partial x^2} \psi_0 = 4a^2 A_0 e^{-ax^2}$$

$$\langle p^2 \rangle = -\hbar^2 \int \psi_0 \frac{\partial^2 \psi_0}{\partial x^2} dx = -\hbar^2 4a^2 A_0^2 \int e^{-2ax^2} dx$$

$$= -4a^2 \hbar^2 (\quad)^2 \qquad =$$

ψ_1: $$\frac{\partial \psi_1}{\partial x} = \frac{\partial}{\partial x} A_1 b x e^{-ax^2} = A_1 b\left(1 + x(-2ax)\right) e^{-ax^2}$$
$$= A_1 b (1 - 2ax^2) e^{-ax^2}$$

$$\frac{\partial^2 \psi_1}{\partial x^2} = A_1 b e^{-ax^2} \left[(1-2ax^2)(-2ax) + (-4ax)\right]$$
$$[4a^2x^3 - 2ax - 4ax] = 4a^2x^3 - 6ax$$

$$\langle p^2 \rangle = -\hbar^2 \int \psi_1 \frac{\partial^2 \psi_1}{\partial x^2} dx = -\hbar^2 \int A_1 b x e^{-ax^2} A_1 b e^{-ax^2} (4a^2x^3 - 6ax) dx$$

$$= -\hbar^2 A_1^2 b^2 \int_{-\infty}^{\infty} e^{-2ax^2} (4a^2 x^4 - 6ax^2) dx$$

$$\int e^{-2ax^2} x^4 dx =$$

$$\int e^{-2ax^2} x^2 dx =$$

(b) Uncertainty principle:

ψ_0: $\langle x^2\rangle - \langle x\rangle^2 = (\Delta x)^2 =$

$\langle p^2\rangle - \langle p\rangle^2 = (\Delta p)^2 = \hbar\omega m$

$\Delta x\,\Delta p = (\qquad)\sqrt{\hbar\omega m} = \qquad \geq \frac{\hbar}{2}$

ψ_1: $\langle x^2\rangle - \langle x\rangle^2 = (\Delta x)^2 =$

$\langle p^2\rangle - \langle p\rangle^2 = (\Delta p)^2 = 3\hbar\omega m$

$\Delta x\,\Delta p = (\qquad)\sqrt{3\hbar\omega m} = \qquad \geq \frac{\hbar}{2}$

(c) $V = \frac{1}{2}kx^2$ where $F = ma = -kx = -m\omega^2 x \rightarrow k = m\omega^2$

$V = \frac{1}{2}m\omega^2 x^2$

$\langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle$ and $\langle T\rangle = \langle p^2\rangle/2m$

ψ_0: $\langle x^2\rangle =$ $\qquad\qquad$ $\langle p^2\rangle =$

so $\langle V\rangle =$

$\langle T\rangle =$

(c) for ψ_1, $\langle x^2 \rangle =$ $\langle p^2 \rangle =$

So $\langle V \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle =$

and $\langle T \rangle = \dfrac{\langle p^2 \rangle}{2m} =$

Are the sums what we expect? Should get

$\langle E \rangle = \langle T \rangle + \langle V \rangle$ where $\langle E \rangle = (n + \frac{1}{2})\hbar\omega$

ψ_0: $\langle E_0 \rangle = \dfrac{\hbar\omega}{2}$, $\langle T \rangle + \langle V \rangle =$

ψ_1: $\langle E_1 \rangle = \dfrac{3\hbar\omega}{2}$, $\langle T \rangle + \langle V \rangle =$