Problem 1.31 Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0, 0, 0)$, $\mathbf{b} = (1, 1, 1)$, and the three paths in Fig. 1.28:

(a) $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1)$;

(b) $(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 1, 1) \rightarrow (1, 1, 1)$;

(c) the parabolic path $z = x^2$; $y = x$.

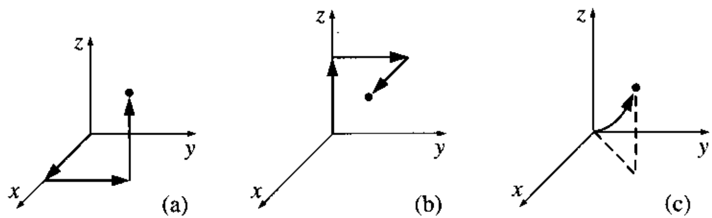

Figure 1.28

1.3.4 The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.} \tag{1.56}$$

In honor, I suppose of its great importance, this theorem has at least three special names: **Gauss's theorem**, **Green's theorem**, or, simply, the **divergence theorem**. Like the other "fundamental theorems," it says that the *integral* of a *derivative* (in this case the *divergence*) clockwise.) Since $x = 0$ for this surface,

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2 \, dy \, dz = \tfrac{4}{3}.$$

Now, what about the line integral? We must break this up into four segments:

$$\begin{array}{lllll}
\text{(i)} & x = 0, & z = 0, & \mathbf{v} \cdot d\mathbf{l} = 3y^2 \, dy, & \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 3y^2 \, dy = 1, \\
\text{(ii)} & x = 0, & y = 1, & \mathbf{v} \cdot d\mathbf{l} = 4z^2 \, dz, & \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4z^2 \, dz = \tfrac{4}{3}, \\
\text{(iii)} & x = 0, & z = 1, & \mathbf{v} \cdot d\mathbf{l} = 3y^2 \, dy, & \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 3y^2 \, dy = -1, \\
\text{(iv)} & x = 0, & y = 0, & \mathbf{v} \cdot d\mathbf{l} = 0, & \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 0 \, dz = 0.
\end{array}$$

So

$$\oint \mathbf{v} \cdot d\mathbf{l} = 1 + \tfrac{4}{3} - 1 + 0 = \tfrac{4}{3}.$$

It checks.

A point of strategy: notice how I handled step (iii). There is a temptation to write $d\mathbf{l} = -dy\,\hat{\mathbf{y}}$ here, since the path goes to the left. You can get away with this, if you insist, by running the integral from $0 \to 1$. Personally, I prefer to say $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$ *always* (never any minus signs) and let the limits of the integral take care of the direction.

Problem 1.33 Test Stokes' theorem for the function $\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}}$, using the triangular shaded area of Fig. 1.34.

Problem 1.34 Check Corollary 1 by using the same function and boundary line as in Ex. 1.11, but integrating over the five sides of the cube in Fig. 1.35. The back of the cube is open.