In $\bar{S}$, meanwhile, it has moved a distance

$$d\bar{x} = \gamma(dx - v\,dt),$$

as we see from (i), in a time given by (iv):

$$d\bar{t} = \gamma\left(dt - \frac{v}{c^2}dx\right).$$

The velocity in $\bar{S}$ is therefore

$$\bar{u} = \frac{d\bar{x}}{d\bar{t}} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - v/c^2 dx\right)} = \frac{(dx/dt - v)}{1 - v/c^2 dx/dt} = \frac{u - v}{1 - uv/c^2}. \tag{12.20}$$

This is **Einstein's velocity addition rule**. To recover the more transparent notation of Eq. 12.3, let A be the particle, B be $\mathcal{S}$, and C be $\bar{\mathcal{S}}$; then $u = v_{AB}$, $\bar{u} = v_{AC}$, and $v = v_{CB} = -v_{BC}$, so Eq. 12.20 becomes

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)},$$

as before.

Problem 12.12 Solve Eqs. 12.18 for x, y, z, t in terms of $\bar{x}, \bar{y}, \bar{z}, \bar{t}$, and check that you recover Eqs. 12.19.

Problem 12.13 Sophie Zabar, clairvoyante, cried out in pain at precisely the instant her twin brother, 500 km away, hit his thumb with a hammer. A skeptical scientist observed both events (brother's accident, Sophie's cry) from an airplane traveling at $\frac{12}{13}c$ to the right (see Fig. 12.19). Which event occurred first, according to the scientist? How *much* earlier was it, in seconds?

Problem 12.14

(a) In Ex. 12.6 we found how velocities *in the x direction* transform when you go from $\mathcal{S}$ to $\bar{\mathcal{S}}$. Derive the analogous formulas for velocities in the y and z directions.

(b) A spotlight is mounted on a boat so that its beam makes an angle $\bar{\theta}$ with the deck (Fig. 12.20). If this boat is then set in motion at speed v, what angle θ does an observer on the *dock* say the beam makes with the deck? Compare Prob. 12.10, and explain the difference.

Problem 12.15 You probably did Prob. 12.4 from the point of view of an observer on the *ground*. Now do it from the point of view of the police car, the outlaws, and the bullet. That is, fill in the gaps in the following table:

speed of → relative to ↓	Ground	Police	Outlaws	Bullet	Do they escape?
Ground	0	$\frac{1}{2}c$	$\frac{3}{4}c$		
Police				$\frac{1}{3}c$	
Outlaws					
Bullet					

which is the inescapable conclusion. There *cannot* be a law of contraction (or expansion) of perpendicular dimensions, for it would lead to irreconcilably inconsistent predictions.

Problem 12.9 A Lincoln Continental is twice as long as a VW Beetle, when they are at rest. As the Continental overtakes the VW, going through a speed trap, a (stationary) policeman observes that they both have the same length. The VW is going at half the speed of light. How fast is the Lincoln going? (Leave your answer as a multiple of c.)

Problem 12.10 A sailboat is manufactured so that the mast leans at an angle $\bar{\theta}$ with respect to the deck. An observer standing on a dock sees the boat go by at speed v (Fig. 12.14). What angle does this *observer* say the mast makes?

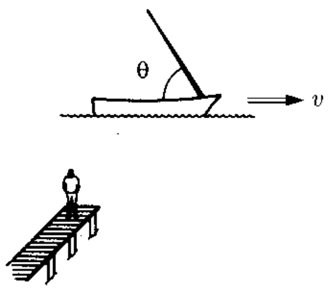

Figure 12.14

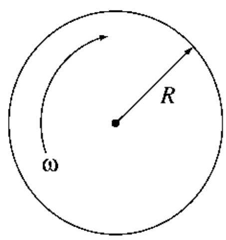

Figure 12.15

! **Problem 12.11** A record turntable of radius R rotates at angular velocity ω (Fig. 12.15). The circumference is presumably Lorentz-contracted, but the radius (being perpendicular to the velocity) is *not*. What's the ratio of the circumference to the diameter, in terms of ω and R? According to the rules of ordinary geometry, that has to be π. What's going on here? [This is known as **Ehrenfest's paradox**; for discussion and references see H. Arzelies, *Relativistic Kinematics*, Chap. IX (Elmsford, NY: Pergamon Press, 1966) and T. A. Weber, *Am. J. Phys.* **65**, 486 (1997).]

12.1.3 The Lorentz Transformations

Any physical process consists of one or more **events**. An "event" is something that takes place at a specific location (x, y, z), at a precise time (t). The explosion of a firecracker, for example, is an event; a tour of Europe is not. Suppose that we know the coordinates (x, y, z) of a particular event E in *one* inertial system $\mathcal{S}$, and we would like to calculate the coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ of that *same event* in some other inertial system $\bar{\mathcal{S}}$. What we need is a "dictionary" for translating from the language of $\mathcal{S}$ to the language of $\bar{\mathcal{S}}$.

Depending on the two events in question, the interval can be positive, negative, or zero:

1. If $I < 0$ we call the interval **timelike**, for this is the sign we get when the two occur at the *same place* ($d = 0$), and are separated only temporally. could be cause!
2. If $I > 0$ we call the interval **spacelike**, for this is the sign we get when the two occur at the *same time* ($t = 0$), and are separated only spatially. not cause!
3. If $I = 0$ we call the interval **lightlike**, for this is the relation that holds when the two events are connected by a signal traveling at the speed of light.

If the interval between the two events is timelike, there exists an inertial system (accessible by Lorentz transformation) in which they occur at the same point. For if I hop on a train going from (A) to (B) at the speed $v = d/t$, leaving event A when it occurs, I shall be just in time to pass B when *it* occurs; in the train system, A and B take place at the same point. You cannot do this for a *spacelike* interval, of course, because v would have to be greater than c, and no observer can exceed the speed of light (γ would be imaginary and the Lorentz transformations would be nonsense). On the other hand, if the interval is spacelike, then there exists a system in which the two events occur at the same time (see Prob. 12.21).

Problem 12.20

(a) Event A happens at point ($x_A = 5$, $y_A = 3$, $z_A = 0$) and at time t_A given by $ct_A = 15$; event B occurs at $(10, 8, 0)$ and $ct_B = 5$, both in system $\mathcal{S}$.

(i) What is the invariant interval between A and B?

(ii) Is there an inertial system in which they occur *simultaneously*? If so, find its velocity (magnitude and direction) relative to $\mathcal{S}$.

(iii) Is there an inertial system in which they occur at the same point? If so, find its velocity relative to $\mathcal{S}$

(b) Repeat part (a) for $A = (2, 0, 0)$, $ct = 1$; and $B = (5, 0, 0)$, $ct = 3$.

Problem 12.21 The coordinates of event A are $(x_A, 0, 0)$, t_A, and the coordinates of event B are $(x_B, 0, 0)$, t_B. Assuming the interval between them is spacelike, find the velocity of the system in which they are simultaneous.

(iii) Space-time diagrams. If you want to represent the motion of a particle graphically, the normal practice is to plot the position versus time (that is, x runs vertically and t horizontally). On such a graph, the velocity can be read off as the slope of the curve. For some reason the convention is reversed in relativity: everyone plots position horizontally and time (or, better, $x^0 = ct$) vertically. Velocity is then given by the *reciprocal* of the slope. A particle at rest is represented by a vertical line; a photon, traveling at the speed of light, is described by a 45° line; and a rocket going at some intermediate speed follows a line of slope $c/v = 1/\beta$ (Fig. 12.21). We call such plots **Minkowski diagrams**.

The trajectory of a particle on a Minkowski diagram is called a **world line**. Suppose you set out from the origin at time $t = 0$. Because no material object can travel faster than light, your world line can never have a slope less than 1. Accordingly, your motion is

Finally, conservation of energy says that

$$\begin{aligned} E_0 + mc^2 &= E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2} \\ &= E + \sqrt{m^2c^4 + E_0^2 - 2E_0E\cos\theta + E^2}. \end{aligned}$$

Solving for E, I find that

$$E = \frac{1}{(1-\cos\theta/mc^2) + (1/E_0)}. \tag{12.57}$$

The answer looks nicer when expressed in terms of photon *wavelength:*

$$E = h\nu = \frac{hc}{\lambda},$$

so

$$\lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta). \tag{12.58}$$

The quantity (h/mc) is called the **Compton wavelength** of the electron.

Problem 12.31 Find the velocity of the muon in Ex. 12.8.

Problem 12.32 A particle of mass m whose total energy is twice its rest energy collides with an identical particle at rest. If they stick together, what is the mass of the resulting composite particle? What is its velocity?

Problem 12.33 A neutral pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ decays into two photons. One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. Find the (relativistic) energy of each photon.

Problem 12.34 In the past, most experiments in particle physics involved stationary targets: one particle (usually a proton or an electron) was accelerated to a high energy E, and collided with a target particle at rest (Fig. 12.29a). Far higher *relative* energies are obtainable (with the same accelerator) if you accelerate *both* particles to energy E, and fire them at each other (Fig. 12.29b). *Classically,* the energy $\bar{E}$ of one particle, relative to the other, is just 4E (why?)—not much of a gain (only a factor of 4). But *relativistically* the gain can be *enormous.* Assuming the two particles have the same mass, m, show that

$$\bar{E} = \frac{2E^2}{mc^2} - mc^2. \tag{12.59}$$

Figure 12.29

as one would certainly expect (after all, the loop as a whole is not moving). But relativistically $\mathbf{p} = \gamma M\mathbf{u}$, and we get

$$p = \gamma_+ M N_+ u_+ - \gamma_- M N_- u_- = \frac{MIl}{Q}(\gamma_+ - \gamma_-),$$

which is *not* zero, because the particles in the upper segment are moving faster.

In fact, the gain in energy (γMc^2), as a particle goes up the left segment, is equal to the work done by the electric force, QEw, where w is the height of the rectangle, so

$$\gamma_+ - \gamma_- = \frac{QEw}{Mc^2},$$

and hence

$$p = \frac{IlEw}{c^2}.$$

But Ilw is the magnetic dipole moment of the loop; as vectors, $\mathbf{m}$ points into the page and $\mathbf{p}$ is to the right, so

$$\mathbf{p} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}).$$

Thus a magnetic dipole in an electric field carries linear momentum, *even though it is not moving!* This so-called **hidden momentum** is strictly relativistic, and purely mechanical; it precisely cancels the electromagnetic momentum stored in the fields (see Ex. 8.3; note that both results can be expressed in the form $p = IV/c^2$).

Problem 12.36 In classical mechanics Newton's law can be written in the more familiar form $\mathbf{F} = m\mathbf{a}$. The relativistic equation, $\mathbf{F} = d\mathbf{p}/dt$, *cannot* be so simply expressed. Show, rather, that

$$\mathbf{F} = \frac{m}{\sqrt{1 - u^2/c^2}}\left[\mathbf{a} + \frac{\mathbf{u}(\mathbf{u} \cdot \mathbf{a})}{c^2 - u^2}\right], \tag{12.73}$$

where $\mathbf{a} \equiv d\mathbf{u}/dt$ is the **ordinary acceleration**.

Problem 12.37 Show that it is possible to outrun a light ray, if you're given a sufficient head start, and your feet generate a constant force.

Problem 12.38 Define **proper acceleration** in the obvious way:

$$\alpha^\mu \equiv \frac{d\eta^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2}. \tag{12.74}$$

(a) Find α^0 and $\boldsymbol{\alpha}$ in terms of $\mathbf{u}$ and $\mathbf{a}$ (the ordinary acceleration).

(b) Express $\alpha_\mu \alpha^\mu$ in terms of $\mathbf{u}$ and $\mathbf{a}$.

(c) Show that $\eta^\mu \alpha_\mu = 0$.

(d) Write the Minkowski version of Newton's second law, Eq. 12.70, in terms of α^μ. Evaluate the invariant product $K^\mu \eta_\mu$.

Problem 12.39 Show that

$$K_\mu K^\mu = \frac{1 - (u^2/c^2)\cos^2\theta}{1 - u^2/c^2} F^2,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{F}$.

Problem 12.40 Show that the (ordinary) acceleration of a particle of mass m and charge q, moving at velocity $\mathbf{u}$ under the influence of electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, is given by

$$\mathbf{a} = \frac{q}{m}\sqrt{1 - u^2/c^2}\left[\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{c^2}\mathbf{u}(\mathbf{u} \cdot \mathbf{E})\right].$$

[*Hint:* Use Eq. 12.73.]

12.3 Relativistic Electrodynamics

12.3.1 Magnetism as a Relativistic Phenomenon

Unlike Newtonian mechanics, classical electrodynamics is *already* consistent with special relativity. Maxwell's equations and the Lorentz force law can be applied legitimately in any inertial system. Of course, what one observer interprets as an electrical process another may regard as magnetic, but the actual particle motions they predict will be identical. To the extent that this did *not* work out for Lorentz and others, who studied the question in the late nineteenth century, the fault lay with the nonrelativistic mechanics they used, not with the electrodynamics. Having corrected Newtonian mechanics, we are now in a position to develop a complete and consistent formulation of relativistic electrodynamics. But I emphasize that we will not be changing the rules of electrodynamics in the slightest—rather, we will be *expressing* these rules in a notation that exposes and illuminates their relativistic character. As we go along, I shall pause now and then to rederive, using the Lorentz transformations, results obtained earlier by more laborious means. But the main purpose of this section is to provide you with a deeper understanding of the structure of electrodynamics—laws that had seemed arbitrary and unrelated before take on a kind of coherence and inevitability when approached from the point of view of relativity.

To begin with I'd like to show you why there *had* to be such a thing as magnetism, given electrostatics and relativity, and how, in particular, you can calculate the magnetic force between a current-carrying wire and a moving charge without ever invoking the laws of magnetism.[14] Suppose you had a string of positive charges moving along to the right at speed v. I'll assume the charges are close enough together so that we may regard them as a continuous line charge λ. Superimposed on this positive string is a negative one, $-\lambda$ proceeding to the left at the same speed v. We have, then, a net current to the right, of magnitude

$$I = 2\lambda v. \tag{12.75}$$

[14]This and several other arguments in this section are adapted from E. M. Purcell's *Electricity and Magnetism*, 2d ed. (New York: McGraw-Hill, 1985).

Problem 12.41 Why can't the electric field in Fig. 12.35b have a z component? After all, the *magnetic* field does.

Problem 12.42 A parallel-plate capacitor, at rest in $\mathcal{S}_0$ and tilted at a 45° angle to the x_0 axis, carries charge densities $\pm\sigma_0$ on the two plates (Fig. 12.41). System $\mathcal{S}$ is moving to the right at speed v relative to $\mathcal{S}_0$.

(a) Find $\mathbf{E}_0$, the field in $\mathcal{S}_0$.

(b) Find $\mathbf{E}$, the field in $\mathcal{S}$.

(c) What angle do the plates make with the x axis?

(d) Is the field perpendicular to the plates in $\mathcal{S}$?

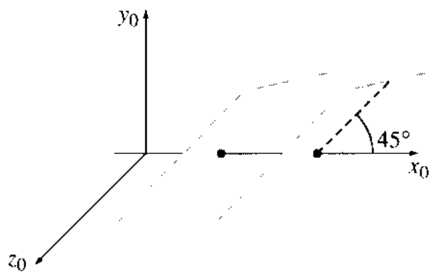

Figure 12.41

Problem 12.43

(a) Check that Gauss's law, $\int \mathbf{E} \cdot d\mathbf{a} = (1/\epsilon_0)Q_{\text{enc}}$, is obeyed by the field of a point charge in uniform motion, by integrating over a sphere of radius R centered on the charge.

(b) Find the Poynting vector for a point charge in uniform motion. (Say the charge is going in the z direction at speed v, and calculate $\mathbf{S}$ at the instant q passes the origin.)

Problem 12.44

(a) Charge q_A is at rest at the origin in system $\mathcal{S}$; charge q_B flies by at speed v on a trajectory parallel to the x axis, but at $y = d$. What is the electromagnetic force on q_B as it crosses the y axis?

(b) Now study the same problem from system $\bar{\mathcal{S}}$, which moves to the right with speed v. What is the force on q_B when q_A passes the $\bar{y}$ axis? [Do it two ways: (i) by using your answer to (a) and transforming the force; (ii) by computing the fields in $\bar{\mathcal{S}}$ and using the Lorentz force law.]

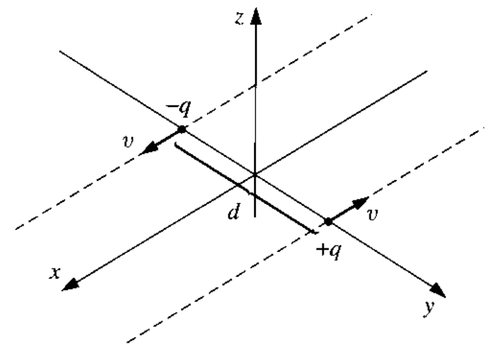

Figure 12.42

Problem 12.45 Two charges $\pm q$, are on parallel trajectories a distance d apart, moving with equal speeds v in opposite directions. We're interested in the force on $+q$ due to $-q$ at the instant they cross (Fig. 12.42). Fill in the following table, doing all the consistency checks you can think of as you go along.

	System A (Fig. 12.42)	System B ($+q$ at rest)	System C ($-q$ at rest)
E at $+q$ due to $-q$:			
B at $+q$ due to $-q$:			
F on $+q$ due to $-q$:			

Problem 12.46

(a) Show that $(\mathbf{E}\cdot\mathbf{B})$ is relativistically invariant.

(b) Show that $(E^2 - c^2B^2)$ is relativistically invariant.

(c) Suppose that in one inertial system $\mathbf{B} = 0$ but $\mathbf{E} \neq 0$ (at some point P). Is it possible to find another system in which the *electric* field is zero at P?

Problem 12.47 An electromagnetic plane wave of (angular) frequency ω is traveling in the x direction through the vacuum. It is polarized in the y direction, and the amplitude of the electric field is E_0.

(a) Write down the electric and magnetic fields, $\mathbf{E}(x, y, z, t)$ and $\mathbf{B}(x, y, z, t)$. [Be sure to define any auxiliary quantities you introduce, in terms of ω, E_0, and the constants of nature.]

(b) This same wave is observed from an inertial system $\bar{\mathcal{S}}$ moving in the x direction with speed v relative to the original system $\mathcal{S}$. Find the electric and magnetic fields in $\bar{\mathcal{S}}$, and express them in terms of the $\bar{\mathcal{S}}$ coordinates: $\bar{\mathbf{E}}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ and $\bar{\mathbf{B}}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$. [Again, be sure to define any auxiliary quantities you introduce.]

(c) What is the frequency $\bar{\omega}$ of the wave in $\bar{\mathcal{S}}$? Interpret this result. What is the wavelength $\bar{\lambda}$ of the wave in $\bar{\mathcal{S}}$? From $\bar{\omega}$ and $\bar{\lambda}$, determine the speed of the waves in $\bar{\mathcal{S}}$. Is it what you expected?