Physical Systems – spring 2007

QM HW 6a, L and Spherical coordinates, Tues.8 May 2007

Shankar **12.3.1, 12.5.3** (i-iii), **12.5.12, 12.5.13, 12.6.1, 12.6.2**

The fact that complex eigenvalues enter the answer, signals that we are overlooking the Hermiticity constraint. Let us impose it. The condition

$$\langle \psi_1 | L_z | \psi_2 \rangle = \langle \psi_2 | L_z | \psi_1 \rangle^* \tag{12.3.4}$$

becomes in the coordinate basis $L|\psi\rangle = -i\hbar \frac{\partial}{\partial \phi} \psi$

$$\int_0^\infty \int_0^{2\pi} \psi_1^* \left(-i\hbar \frac{\partial}{\partial \phi}\right) \psi_2 \varrho \, d\varrho \, d\phi = \left[\int_0^\infty \int_0^{2\pi} \psi_2^* \left(-i\hbar \frac{\partial}{\partial \phi}\right) \psi_1 \varrho \, d\varrho \, d\phi \right]^*$$

$$= \int \int \psi_2 \left(i\hbar \frac{\partial}{\partial \phi}\right) \psi^* \rho \, d\rho \, d\phi \tag{12.3.5}$$

If this requirement is to be satisfied for all ψ_1 and ψ_2, one can show (upon integrating by parts) that each ψ must obey

Hermiticity → periodicity : $$\psi(\varrho, 0) = \psi(\varrho, 2\pi) \tag{12.3.6}$$

If we impose this constraint on the L_z eigenfunctions, Eq. (12.3.3), we find

$$e^0 = 1 = e^{2\pi i l_z/\hbar} \tag{12.3.7}$$

This forces l_z not merely to be real, but also an integral multiple of $\hbar$:

→ QUANTIZATION! $$\boxed{l_z = m\hbar,} \quad m = 0, \pm 1, \pm 2, \ldots \tag{12.3.8}$$

One calls m the *magnetic quantum number*. Notice that $l_z = m\hbar$ implies that ψ is a single-valued function of ϕ.

#1 *Exercise 12.3.1.* Provide the steps linking Eq. (12.3.5) to Eq. (12.3.6).

2 *Exercise 12.5.3.** (i) Show that $\langle J_x \rangle = \langle J_y \rangle = 0$ in a state $| jm \rangle$.

(ii) Show that in these states

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = \tfrac{1}{2}\hbar^2 [j(j+1) - m^2]$$

(use symmetry arguments to relate $\langle J_x^2 \rangle$ to $\langle J_y^2 \rangle$).

(iii) Check that $\Delta J_x \cdot \Delta J_y$ from part (ii) satisfies the inequality imposed by the uncertainty principle [Eq. (9.2.9)]. $(\Delta J_x)^2 (\Delta J_y)^2 \geq |\langle \psi | J_x J_y | \psi \rangle|^2$

*Exercise 12.5.12.** Since L^2 and L_z commute with Π, they should share a basis with it. Verify that $Y_l^m \xrightarrow[\Pi]{} (-1)^l Y_l^m$. (First show that $\theta \to \pi - \theta, \phi \to \phi + \pi$ under parity. Prove the result for Y_l^l. Verify that L_- does not alter the parity, thereby proving the result for all Y_l^m.)

*Exercise 12.5.13.** Consider a particle in a state described by

$$\psi = N(x + y + 2z)e^{-\alpha r}$$ Ex 12.5.10 (ii)?

where N is a normalization factor.

(i) Show, by rewriting the $Y_1^{\pm 1, 0}$ functions in terms of x, y, z, and r, that

$$\begin{aligned} Y_1^{\pm 1} &= \mp \left(\frac{3}{4\pi}\right)^{1/2} \frac{x \pm iy}{2^{1/2} r} \\ Y_1^0 &= \left(\frac{3}{4\pi}\right)^{1/2} \frac{z}{r} \end{aligned} \tag{12.5.42}$$

(ii) Using this result, show that for a particle described by ψ above, $P(l_z = 0) = 2/3$; $P(l_z = +\hbar) = 1/6 = P(l_z = -\hbar)$. Hint: Expand $\psi(r)$ in terms of the Y_l^m's.

Here are the first few Y_l^m functions:

$$\begin{aligned} Y_0^0 &= (4\pi)^{-1/2} \\ Y_1^{\pm 1} &= \mp (3/8\pi)^{1/2} \sin\theta \, e^{\pm i\phi} \\ Y_1^0 &= (3/4\pi)^{1/2} \cos\theta \\ Y_2^{\pm 2} &= (15/32\pi)^{1/2} \sin^2\theta \, e^{\pm 2i\phi} \\ Y_2^{\pm 1} &= \mp (15/8\pi)^{1/2} \sin\theta \cos\theta \, e^{\pm i\phi} \\ Y_2^0 &= (5/16\pi)^{1/2} (3\cos^2\theta - 1) \end{aligned} \tag{12.5.39}$$

Note that

$$Y_l^{-m} = (-1)^m (Y_l^m)^* \tag{12.5.40}$$

4.1 *Exercise 12.6.1.** A particle is described by the wave function

$$\psi_E(r, \theta, \phi) = A e^{-r/a_0} \qquad (a_0 = \text{const})$$

(i) What is the angular momentum content of the state?

(ii) Assuming ψ_E is an eigenstate in a potential that vanishes as $r \to \infty$, find E. (Match leading terms in Schrödinger's equation.) $E = -\hbar^2/2\mu a_0^2$

(iii) Having found E, consider finite r and find $V(r)$. $V = \hbar^2/\mu a_0 r$

*Exercise 12.6.2.** Provide the steps connecting Eq. (12.6.3) and Eq. (12.6.5).

12.6. Solution of Rotationally Invariant Problems

Spherically S

We now consider a class of problems of great practical interest: problems where $V(r, \theta, \phi) = V(r)$. The Schrödinger equation in spherical coordinates becomes

$dV = \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$

$$\left[\frac{-\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)+V(r)\right] \times \psi_E(r,\theta,\phi) = E\psi_E(r,\theta,\phi) \qquad (12.6.1)$$

Since $[H, \mathbf{L}] = 0$ for a spherically symmetric potential, we seek simultaneous eigenfunctions of H, L^2, and L_z:

$$\psi_{Elm}(r,\theta,\phi) = R_{Elm}(r)Y_l^m(\theta,\phi) \qquad (12.6.2)$$

Feeding in this form, and bearing in mind that the angular part of ∇^2 is just the L^2 operator in the coordinate basis [up to a factor $(-\hbar^2 r^2)^{-1}$, see Eq. (12.5.36)], we get the *radial equation* p. 324

$$\left\{-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{l(l+1)}{r^2}\right]+V(r)\right\}R_{El} = ER_{El} \qquad (12.6.3)$$

Notice that the subscript m has been dropped: neither the energy for the radial function depends on it. We find, as anticipated earlier, the $(2l+1)$-fold degeneracy of H.

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At this point it becomes fruitful to introduce an auxiliary function U_{El} defined as follows:

rotation operator?

$$R_{El} = U_{El}/r \qquad (12.6.4)$$

RADIAL WAVEFUNCTION

and which obeys the equation

$$\left\{\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{l(l+1)\hbar^2}{2\mu r^2}\right]\right\}U_{El} = 0 \qquad (12.6.5)$$