Physical Systems – spring 2007

QM HW 2a Tues.10 April 2007 - Moore SP4.1, Shankar ~~4.2.1~~ (p.135, ed.1), 4.2.2 (p.146), 4.2.3

Instead of 4.2.1, do Moore's very similar and equally important (but less tedious) problem:

Moore SP4.1: The following operators are the Hermitian operators corresponding to the x,y,z components of a fermion's spin angular momentum respectively (as we will see in a later chapter):

$$S_x \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ S_y \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ S_z \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that we have expressed these matrices in the basis of the eigenstates of S_z.

a. If we measure the z-component of the particle's spin, what are the possible results that we could get?

b. Take the state where $s_z = \hbar/2$ (i.e. the eigenstate of S_z with eigenvalue $s_z = \hbar/2$).
In this state, what are $\langle S_x \rangle, \langle S_x^2 \rangle,\ and\ \Delta S_x$?

c. Find the eigenvalues and normalized eigenstates of S_x in the S_z basis (i.e. the basis currently being used).

d. If the particle is in the state with $s_z = -\ \hbar/2$ and s_x is measured, what are the possible outcomes and their probabilities?

e. Consider the state $|\psi\rangle \leftrightarrow \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$, which we have expressed in the same basis we have been using all along, the basis of eigenstates of S_z. If s_z^2 is measured and $\hbar^2/4$ is the result obtained, what is the state after the measurement?
How probable was this result?
If s_z is then measured, what are the possible outcomes and their respective prabablities?

f. A particle is in a state with probabilities P($s_z = \hbar/2$) = 2/5 and P($s_z = -\ \hbar/2$) = 3/5.
Convince yourself that the most general normalized state with this property is:

$$|\psi\rangle = \tfrac{2}{5} e^{i\delta_1} \left| s_z = \tfrac{\hbar}{2} \right\rangle + \tfrac{3}{5} e^{i\delta_2} \left| s_z = \tfrac{-\hbar}{2} \right\rangle .$$

If $|\psi\rangle$ is a normalized state, then the state $e^{i\theta}|\psi\rangle$ is a *physically equivalent* normalized state. So the overall phase θ of the state above is physically unimportant. What about the relative phase $e^{i(\delta_1-\delta_2)}$ between the two parts of the state?
[Calculate, for example, P($s_x = \hbar/2$) for the state above.]

*Exercise 4.2.2.** Show that for a real wave function $\psi(x)$, the expectation value of momentum $\langle P \rangle = 0$. (Hint: Show that the probabilities for the momenta $\pm p$ are equal). Generalize this result to the case $\psi = c\psi_r$, where ψ_r is real and c an arbitrary (real or complex) constant. (Recall that $|\psi\rangle$ and $\alpha|\psi\rangle$ are physically equivalent.)

*Exercise 4.2.3.** Show that if $\psi(x)$ has mean momentum $\langle P \rangle$, $e^{ip_0x/\hbar}\psi(x)$ has mean momentum $\langle P \rangle + p_0$.

QM HW 2b Thus.12 April 2007 - **Shankar 5.1.2** (p.161, ed.1), **5.2.1** (p.163), **5.3.2** (p.176. Why? See 4.2.2)

*Exercise 5.1.2.** By solving the eigenvalue equation (5.1.3) in the X basis, regain Eq. (5.1.8), i.e., show that the general solution of energy E is

$$\psi_E(x) = \beta \frac{\exp[i(2mE)^{1/2}x/\hbar]}{(2\pi\hbar)^{1/2}} + \gamma \frac{\exp[-i(2mE)^{1/2}x/\hbar]}{(2\pi\hbar)^{1/2}}$$

[The factor $(2\pi\hbar)^{-1/2}$ is arbitrary and may be absorbed into β and γ.] Though $\psi_E(x)$ will satisfy the equation even if $E < 0$, are these functions in the Hilbert space?

$$H\,|E\rangle = \frac{P^2}{2m}\,|E\rangle = E\,|E\rangle \quad / \text{ eigenvalue eqn} \tag{5.1.3}$$

$$|E\rangle = \beta\,|p = (2mE)^{1/2}\rangle + \gamma\,|p = -(2mE)^{1/2}\rangle \tag{5.1.8}$$

*Exercise 5.2.1.** A particle is in the ground state of a box of length L. Suddenly the box expands (symmetrically) to twice its size, leaving the wave function undisturbed. Show that the probability of finding the particle in the ground state of the new box is $(8/3\pi)^2$.

Exercise 5.3.2. Convince yourself that if $\psi = c\tilde{\psi}$, where c is constant (real or complex) and $\tilde{\psi}$ is real, the corresponding j vanishes.