*Exercise 7.3.1.** Consider the question why we tried a power-series solution for Eq. (7.3.11) but not Eq. (7.3.8). By feeding in a series into the latter, verify that a three-term recursion relation between C_{n+2}, C_n, and C_{n-2} obtains, from which the solution does not follow so readily. The problem is that ψ'' has two powers of y less than $2\varepsilon\psi$, while the $-y^2$ piece has two more powers of y. In Eq. (7.3.11) on the other hand, of the three pieces u'', $-2yu'$, and $(2\varepsilon - 1)y$, the last two have the same powers of y.

Exercise 7.3.2. Verify that $H_3(y)$ and $H_4(y)$ obey the recursion relation, Eq. (7.3.15).

Exercise 7.3.3. If $\psi(x)$ is even and $\phi(x)$ is odd under $x \to -x$, show that

$$\int_{-\infty}^{\infty} \psi(x)\phi(x)\,dx = 0$$

Use this to show that $\psi_2(x)$ and $\psi_1(x)$ are orthogonal. Using the table of Gaussian integrals in Appendix A.2 verify that $\psi_2(x)$ and $\psi_0(x)$ are orthogonal.

*Exercise 7.3.4.** Using Eqs. (7.3.23)–(7.3.25), show that

$$\langle n' \mid X \mid n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2} [\delta_{n',n+1}(n+1)^{1/2} + \delta_{n',n-1}n^{1/2}]$$

$$\langle n' \mid P \mid n\rangle = \left(\frac{m\omega\hbar}{2}\right)^{1/2} i[\delta_{n',n+1}(n+1)^{1/2} - \delta_{n',n-1}n^{1/2}]$$

*Exercise 7.3.5.** Using the symmetry arguments from Exercise 7.3.3 show that $\langle n \mid X \mid n\rangle = \langle n \mid P \mid n\rangle = 0$ and thus that $\langle X^2\rangle = (\Delta X)^2$ and $\langle P^2\rangle = (\Delta P)^2$ in these states. Show that $\langle 1 \mid X^2 \mid 1\rangle = 3\hbar/2m\omega$ and $\langle 1 \mid P^2 \mid 1\rangle = \frac{3}{2}m\omega\hbar$. Show that $\psi_0(x)$ saturates the uncertainty bound $\Delta X \cdot \Delta P \geq \hbar/2$.

*Exercise 7.3.6.** Consider a particle in a potential

$$V(x) = \tfrac{1}{2}m\omega^2x^2, \qquad x > 0$$
$$= \infty, \qquad x \leq 0$$

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What are the boundary conditions on the wave functions now? Find the eigenvalues and eigenfunctions.

We now discuss the eigenvalues and eigenfunction of the oscillator. The following are the main features:

(*i*) *The energy is quantized.* In contrast to the classical oscillator whose energy is continuous, the quantum oscillator has a discrete set of levels given by Eq. (7.3.20). Note that the quantization emerges only after we supplement Schrödinger's equation with the requirement that ψ be an

Given the commutation relations between X and P, the ones among dependent operators follow from the repeated use of the relations

$$[\Omega, \Lambda\Gamma] = \Lambda[\Omega, \Gamma] + [\Omega, \Lambda]\Gamma$$

and

$$[\Omega\Lambda, \Gamma] = \Omega[\Lambda, \Gamma] + [\Omega, \Gamma]\Lambda$$

Since PB obey similar rules (Exercise 2.7.1) except for the lack of emphasis on ordering of the classical variables, it turns out that if

$$\{\omega(x, p), \lambda(x, p)\} = \gamma(x, p)$$

then

$$[\Omega(X, P), \Lambda(X, P)] = i\hbar\Gamma(X, P) \tag{7.4.40}$$

except for differences arising from ordering ambiguities; hence the formal similarity between classical and quantum mechanics, first encountered in Chapter 6.

Although the new form of postulate II provides a general, basis-independent specification of the quantum operators corresponding to classical variables, that is to say for "quantizing," in practice one typically works in the X basis and also ignores the latitude in the choice of P_i and sticks to the traditional one, $P_i = -i\hbar\, \partial/\partial x_i$, which leads to the simplest differential equations. The solution to the oscillator problem, given just the commutation relations (and a little help from Dirac) is atypical.

*Exercise 7.4.1.** Compute the matrix elements of X and P in the $|n\rangle$ basis and compare with the result from Exercise 7.3.4.

*Esercise 7.4.2.** Find $\langle X\rangle$, $\langle P\rangle$, $\langle X^2\rangle$, $\langle P^2\rangle$, $\Delta X \cdot \Delta P$ in the state $|n\rangle$.

*Exercise 7.4.3** (*Virial Theorem*). The virial theorem in classical mechanics states that for a particle bound by a potential $V(r) = ar^k$, the average (over the orbit) kinetic and potential energies are related by

$$\bar{T} = c(k)\bar{V}$$

when $c(k)$ depends only on k. Show that $c(k) = k/2$ by considering a circular orbit. Using the results from the previous exercise show that for the oscillator $(k = 2)$

$$\langle T\rangle = \langle V\rangle$$

in the quantum state $|n\rangle$.

Exercise 7.4.4. Show that $\langle n | X^4 | n\rangle = (\hbar/2m\omega)^2[3 + 6n(n + 1)]$.

By projecting the equation

$$|n\rangle = \frac{(a^\dagger)^n}{(n!)^{1/2}}|0\rangle$$

onto the X basis, we get the *normalized* eigenfunctions

$$\langle x|n\rangle = \psi_n\left[x = \left(\frac{\hbar}{m\omega}\right)^{1/2} y\right] = \frac{1}{(n!)^{1/2}}\left[\frac{1}{2^{1/2}}\left(y - \frac{d}{dy}\right)\right]^n \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-y^2/2} \tag{7.5.7}$$

A comparison of the above result with Eq. (7.3.22) shows that

$$H_n(y) = e^{y^2/2}\left(y - \frac{d}{dy}\right)^n e^{-y^2/2} \tag{7.5.8}$$

We now conclude our rather lengthy discussion of the oscillator. If you understand this chapter thoroughly, you should have a good grasp of how quantum mechanics works.

Exercise 7.5.1. Project Eq. (7.5.1) on the P basis and obtain $\psi_0(p)$.

Exercise 7.5.2. Project the relation

$$a|n\rangle = n^{1/2}|n-1\rangle$$

on the X basis and derive the recursion relation

$$H_n'(y) = 2nH_{n-1}(y)$$

using Eq. (7.3.22).

Exercise 7.5.3. Starting with

$$a + a^\dagger = 2^{1/2}y$$

and

$$(a + a^\dagger)|n\rangle = n^{1/2}|n-1\rangle + (n+1)^{1/2}|n+1\rangle$$

and Eq. (7.3.22), derive the relation

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y)$$

Exercise 7.5.4 Thermodynamics of Oscillators.* The Boltzman formula

$$P(i) = e^{-\beta E(i)}/Z$$

where

$$Z = \sum_i e^{-\beta E(i)}$$

*Exercise 7.4.5.** At $t = 0$ a particle starts out in $|\psi(0)\rangle = 1/2^{1/2}(|0\rangle + |1\rangle)$. (i) Find $|\psi(t)\rangle$; (ii) find $\langle X(0)\rangle = \langle\psi(0)|X|\psi(0)\rangle$, $\langle P(0)\rangle$, $\langle X(t)\rangle$, $\langle P(t)\rangle$; (iii) find $\langle\dot{X}(t)\rangle$ and $\langle\dot{P}(t)\rangle$ using Ehrenfest's theorem and solve for $\langle X(t)\rangle$ and $\langle P(t)\rangle$ and compare with part (ii). p.140

*Exercise 7.4.6.** Show that $\langle a(t)\rangle = e^{-i\omega t}\langle a(0)\rangle$ and that $\langle a^\dagger(t)\rangle = e^{i\omega t} \times \langle a^\dagger(0)\rangle$.

Exercise 7.4.7. Verify Eq. (7.4.40) for the case

(i) $\Omega = X, \quad \Lambda = X^2 + P^2$

(ii) $\Omega = X^2, \ \Lambda = P^2$

The second case illustrates the ordering ambiguity.

*Exercise 7.4.8.** Consider the three angular momentum variables in classical mechanics:

$$l_x = yp_z - zp_y$$
$$l_y = zp_x - xp_z$$
$$l_z = xp_y - yp_x$$

(i) Construct L_x, L_y, and L_z, the quantum counterparts, and note that there are no ordering ambiguities.

(ii) Verify that $\{l_x, l_y\} = l_z$ [see Eq. (2.7.3) for the definition of the PB].

(iii) Verify that $[L_x, L_y] = i\hbar L_z$.

Exercise 7.4.9 (Important). Consider the unconventional (but fully acceptable) operator choice

$$X \to x$$
$$P \to -i\hbar\frac{d}{dx} + f(x)$$

in the X basis.

(i) Verify that the canonical commutation relation is satisfied.

(ii) It is possible to interpret the change in the operator assignment as a result of a unitary change of the X basis:

$$|x\rangle \to |\tilde{x}\rangle = e^{ig(X)/\hbar}|x\rangle = e^{ig(x)/\hbar}|x\rangle$$

where

$$g(x) = \int^x f(x')\,dx'$$

First verify that

$$\langle\tilde{x}|X|\tilde{x}'\rangle = x\delta(x - x')$$

i.e.,

$$X \xrightarrow[\text{new } X \text{ basis}]{} x$$

as it should be. But that's not what we're interested in right now. The *momentum* in the fields is

$$\mathbf{p}_{\text{em}} = \mu_0\epsilon_0 \int \mathbf{S}\,d\tau = \frac{\mu_0\lambda I}{4\pi^2}\,\hat{\mathbf{z}}\int_a^b \frac{1}{s^2} l 2\pi s\,ds = \frac{\mu_0\lambda I l}{2\pi}\ln(b/a)\,\hat{\mathbf{z}}.$$

This is an astonishing result. The cable is not moving, and the fields are static, and yet we are asked to believe that there is momentum in the system. If something tells you this cannot be the whole story, you have sound intuitions. In fact, if the center of mass of a localized system is at rest, its total momentum *must* be zero. In this case it turns out that there is "hidden" mechanical momentum associated with the flow of current, and this exactly cancels the momentum in the fields. But *locating* the **hidden momentum** is not easy, and it is actually a relativistic effect, so I shall save it for Chapter 12 (Ex. 12.12).

Suppose now that we turn up the resistance, so the current decreases. The changing magnetic field will induce an electric field (Eq. 7.19):

$$\mathbf{E} = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{\mathbf{z}}.$$

This field exerts a force on $\pm\lambda$:

$$\mathbf{F} = \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln a + K\right]\hat{\mathbf{z}} - \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln b + K\right]\hat{\mathbf{z}} = -\frac{\mu_0\lambda l}{2\pi}\frac{dI}{dt}\ln(b/a)\,\hat{\mathbf{z}}.$$

The total momentum imparted to the cable, as the current drops from I to 0, is therefore

$$\mathbf{p}_{\text{mech}} = \int \mathbf{F}\,dt = \frac{\mu_0\lambda I l}{2\pi}\ln(b/a)\,\hat{\mathbf{z}},$$

which is precisely the momentum originally stored in the fields. (The cable will not recoil, however, because an equal and opposite impulse is delivered by the simultaneous disappearance of the hidden momentum.)

Problem 8.5 Consider an infinite parallel-plate capacitor, with the lower plate (at $z = -d/2$) carrying the charge density $-\sigma$, and the upper plate (at $z = +d/2$) carrying the charge density $+\sigma$.

(a) Determine all nine elements of the stress tensor, in the region between the plates. Display your answer as a 3×3 matrix:

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

(b) Use Eq. 8.22 to determine the force per unit area on the top plate. Compare Eq. 2.51.

(c) What is the momentum per unit area, per unit time, crossing the xy plane (or any other plane parallel to that one, between the plates)?

(d) At the plates this momentum is absorbed, and the plates recoil (unless there is some nonelectrical force holding them in position). Find the recoil force per unit area on the top

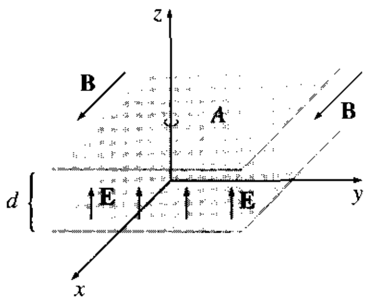

Figure 8.6

plate, and compare your answer to (b). [*Note:* This is not an *additional* force, but rather an alternative way of calculating the *same* force—in (b) we got it from the force law, and in (d) we did it by conservation of momentum.]

Problem 8.6 A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\,\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{x}}$, as shown in Fig. 8.6.[3]

(a) Find the electromagnetic momentum in the space between the plates.

(b) Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge?

(c) Instead of turning off the *electric* field (as in (b)), suppose we slowly reduce the *magnetic* field. This will induce a Faraday electric field, which in turn exerts a force on the plates. Show that the total impulse is (again) equal to the momentum originally stored in the fields.

8.2.4 Angular Momentum

By now the electromagnetic fields (which started out as mediators of forces between charges) have taken on a life of their own. They carry *energy* (Eq. 8.13)

$$u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right), \tag{8.32}$$

and *momentum* (Eq. 8.30)

$$\boldsymbol{\wp}_{\text{em}} = \mu_0\epsilon_0 \mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}), \tag{8.33}$$

and, for that matter, *angular* momentum:

$$\boldsymbol{\ell}_{\text{em}} = \mathbf{r}\times\boldsymbol{\wp}_{\text{em}} = \epsilon_0\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]. \tag{8.34}$$

[3] See F. S. Johnson, B. L. Cragin, and R. R. Hodges, *Am. J. Phys.* **62**, 33 (1994).