what is the subsequent form, $f(z,t)$? Evidently, the displacement at point z, at the later time t, is the same as the displacement a distance vt to the left (i.e. at $z - vt$), back at time $t = 0$:

$$f(z,t) = f(z - vt, 0) = g(z - vt). \tag{9.1}$$

That statement captures (mathematically) the essence of wave motion. It tells us that the function $f(z,t)$, which *might* have depended on z and t in *any* old way, in *fact* depends on them only in the very special combination $z - vt$; when that is true, the function $f(z,t)$ represents a wave of fixed shape traveling in the z direction at speed v. For example, if A and b are constants (with the appropriate units),

$$f_1(z,t) = Ae^{-b(z-vt)^2}, \quad f_2(z,t) = A\sin[b(z - vt)], \quad f_3(z,t) = \frac{A}{b(z-vt)^2 + 1}$$

all represent waves (with different shapes, of course), but

$$f_4(z,t) = Ae^{-b(bz^2+vt)}, \quad \text{and} \quad f_5(z,t) = A\sin(bz)\cos(bvt)^3,$$

do *not*.

Why does a stretched string support wave motion? Actually, it follows from Newton's second law. Imagine a very long string under tension T. If it is displaced from equilibrium, the net transverse force on the segment between z and $z + \Delta z$ (Fig. 9.2) is

$$\Delta F = T\sin\theta' - T\sin\theta,$$

where θ' is the angle the string makes with the z-direction at point $z + \Delta z$, and θ is the corresponding angle at point z. Provided that the distortion of the string is not too great, these angles are small (the figure is exaggerated, obviously), and we can replace the sine by the tangent:

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z.$$

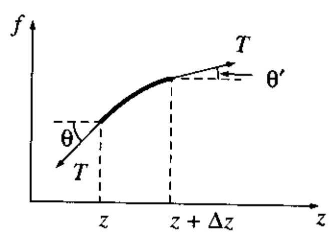

Figure 9.2

surprising is that the *most general* solution to the wave equation is the sum of a wave to the right and a wave to the left:

$$f(z,t) = g(z - vt) + h(z + vt). \tag{9.6}$$

(Notice that the wave equation is **linear**: The sum of any two solutions is itself a solution.) *Every* solution to the wave equation can be expressed in this form.

Like the simple harmonic oscillator equation, the wave equation is ubiquitous in physics. If something is vibrating, the oscillator equation is almost certainly responsible (at least, for small amplitudes), and if something is waving (whether the context is mechanics or acoustics, optics or oceanography), the wave equation (perhaps with some decoration) is bound to be involved.

Problem 9.1 By explicit differentiation, check that the functions f_1, f_2, and f_3 in the text satisfy the wave equation. Show that f_4 and f_5 do *not*.

Problem 9.2 Show that the **standing wave** $f(z,t) = A\sin(kz)\cos(kvt)$ satisfies the wave equation, and express it as the sum of a wave traveling to the left and a wave traveling to the right (Eq. 9.6).

9.1.2 Sinusoidal Waves

(i) Terminology. Of all possible wave forms, the sinusoidal one

$$f(z,t) = A\cos[k(z - vt) + \delta] \tag{9.7}$$

is (for good reason) the most familiar. Figure 9.3 shows this function at time $t = 0$. A is the **amplitude** of the wave (it is positive, and represents the maximum displacement from equilibrium). The argument of the cosine is called the **phase**, and δ is the **phase constant** (obviously, you can add any integer multiple of 2π to δ without changing $f(z,t)$; ordinarily, one uses a value in the range $0 \le \delta < 2\pi$). Notice that at $z = vt - \delta/k$, the phase is zero; let's call this the "central maximum." If $\delta = 0$, the central maximum passes the origin at time $t = 0$; more generally, δ/k is the distance by which the central maximum (and

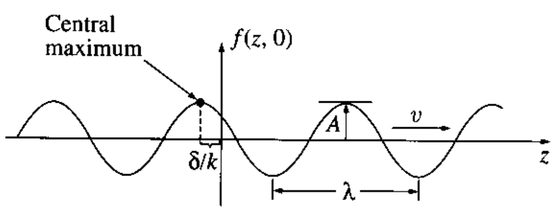

Figure 9.3

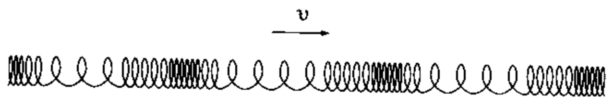

Figure 9.7

Now there are, of course, *two* dimensions perpendicular to any given line of propagation. Accordingly, transverse waves occur in two independent states of **polarization**: you can shake the string up-and-down ("vertical" polarization—Fig. 9.8a),

$$\tilde{\mathbf{f}}_v(z, t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{x}}, \tag{9.34}$$

or left-and-right ("horizontal" polarization—Fig. 9.8b),

$$\tilde{\mathbf{f}}_h(z, t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{y}}, \tag{9.35}$$

or along any other direction in the xy plane (Fig. 9.8c):

$$\tilde{\mathbf{f}}(z, t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{n}}. \tag{9.36}$$

The **polarization vector** $\hat{\mathbf{n}}$ defines the plane of vibration.[2] Because the waves are transverse, $\hat{\mathbf{n}}$ is perpendicular to the direction of propagation:

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = 0. \tag{9.37}$$

In terms of the **polarization angle** θ,

$$\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}. \tag{9.38}$$

Thus, the wave pictured in Fig. 9.8c can be considered a superposition of two waves—one horizontally polarized, the other vertically:

$$\tilde{\mathbf{f}}(z, t) = (\tilde{A}\cos\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{x}} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{y}}. \tag{9.39}$$

Problem 9.8 Equation 9.36 describes the most general **linearly** polarized wave on a string. Linear (or "plane") polarization (so called because the displacement is parallel to a fixed vector $\hat{\mathbf{n}}$) results from the combination of horizontally and vertically polarized waves of the *same phase* (Eq. 9.39). If the two components are of equal amplitude, but *out of phase* by 90° (say, $\delta_v = 0$, $\delta_h = 90°$), the result is a *circularly* polarized wave. In that case:

(a) At a fixed point z, show that the string moves in a circle about the z axis. Does it go *clockwise* or *counterclockwise*, as you look down the axis toward the origin? How would you construct a wave circling the *other* way? (In optics, the clockwise case is called **right circular polarization**, and the counterclockwise, **left circular polarization**.)

(b) Sketch the string at time $t = 0$.

(c) How would you shake the string in order to produce a circularly polarized wave?

---

[2]Notice that you can always switch the *sign* of $\hat{\mathbf{n}}$, provided you simultaneously advance the phase constant by 180°, since both operations change the sign of the wave.