Convective derivative - Chen - Plasma Physics

velocity **v**. The fluid equation is obtained simply by multiplying Eq. [3-29] by the density n:

$$mn\frac{d\mathbf{u}}{dt} = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{3-30}$$

This is, however, not a convenient form to use. In Eq. [3-29], the time derivative is to be taken *at the position of the particles.* On the other hand, we wish to have an equation for fluid elements *fixed in space*, because it would be impractical to do otherwise. Consider a drop of cream in a cup of coffee as a fluid element. As the coffee is stirred, the drop distorts into a filament and finally disperses all over the cup, losing its identity. A fluid element at a fixed spot in the cup, however, retains its identity although particles continually go in and out of it.

To make the transformation to variables in a fixed frame, consider $\mathbf{G}(x, t)$ to be any property of a fluid in one-dimensional x space. The change of **G** with time *in a frame moving with the fluid* is the sum of two terms:

$$\frac{d\mathbf{G}(x,t)}{dt} = \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \mathbf{G}}{\partial x}\frac{dx}{dt} = \frac{\partial \mathbf{G}}{\partial t} + u_x\frac{\partial \mathbf{G}}{\partial x} \tag{3-31}$$

The first term on the right represents the change of **G** at a fixed point in space, and the second term represents the change of **G** as the observer moves with the fluid into a region in which **G** is different. In three dimensions, Eq. [3-31] generalizes to

$$\frac{d\mathbf{G}}{dt} = \frac{\partial \mathbf{G}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{G} \tag{3-32}$$

This is called the *convective derivative* and is sometimes written $D\mathbf{G}/Dt$. Note that $(\mathbf{u} \cdot \nabla)$ is a *scalar* differential operator. Since the sign of this term is sometimes a source of confusion, we give two simple examples.

Figure 3-1 shows an electric water heater in which the hot water has risen to the top and the cold water has sunk to the bottom. Let $G(x, t)$ be the temperature T; ∇G is then upward. Consider a fluid element near the edge of the tank. If the heater element is turned on, the fluid element is heated as it moves, and we have $dT/dt > 0$. If, in addition, a paddle wheel sets up a flow pattern as shown, the temperature in a *fixed* fluid element is lowered by the convection of cold water from the bottom. In this case, we have $\partial T/\partial x > 0$ and $u_x > 0$, so that $\mathbf{u} \cdot \nabla T > 0$. The temperature change in the fixed element, $\partial T/\partial t$, is given by a balance

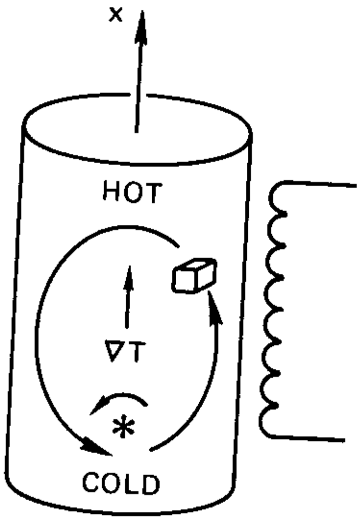


FIGURE 3-1 **Motion of fluid elements in a hot water heater.**

of these effects,

$$\frac{\partial T}{\partial t} = \frac{dT}{dt} - \mathbf{u} \cdot \nabla T \qquad [3\text{-}33]$$

It is quite clear that $\partial T/\partial t$ can be made zero, at least for a short time.

As a second example we may take G to be the salinity S of the water near the mouth of a river (Fig. 3-2). If x is the upstream direction, there

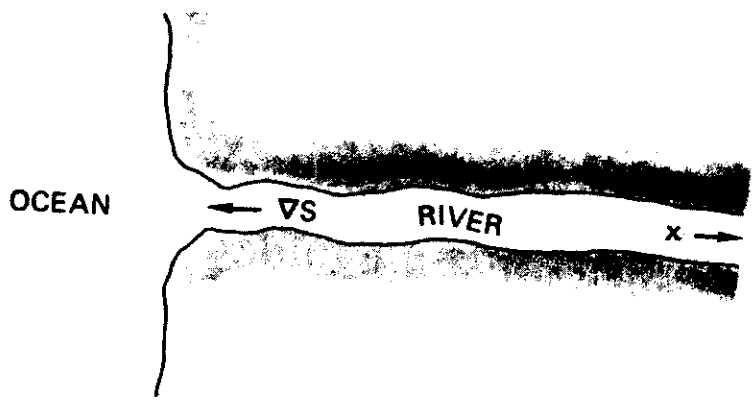


FIGURE 3-2 **Direction of the salinity gradient at the mouth of a river.**

is normally a gradient of S such that $\partial S/\partial x < 0$. When the tide comes in, the entire interface between salt and fresh water moves upstream, and $u_x > 0$. Thus

$$\frac{\partial S}{\partial t} = -u_x \frac{\partial S}{\partial x} > 0 \qquad [3\text{-}34]$$

meaning that the salinity increases at any given point. Of course, if it rains, the salinity decreases everywhere, and a negative term dS/dt is to be added to the middle part of Eq. [3-34].

As a final example, take G to be the density of cars near a freeway entrance at rush hour. A driver will see the density around him increasing as he approaches the crowded freeway. This is the convective term $(\mathbf{u} \cdot \nabla)G$. At the same time, the local streets may be filling with cars that enter from driveways, so that the density will increase even if the observer does not move. This is the $\partial G/\partial t$ term. The total increase seen by the observer is the sum of these effects.

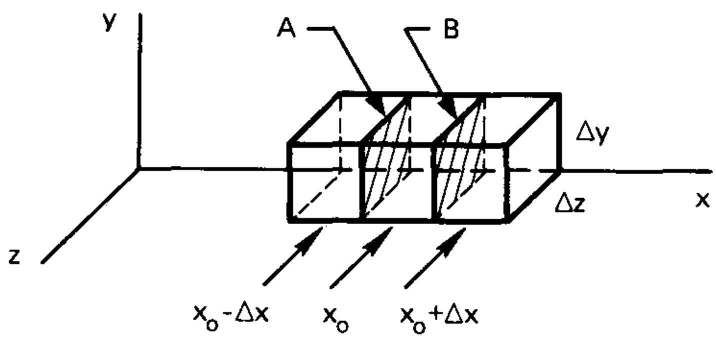
In the case of a plasma, we take $\mathbf{G}$ to be the fluid velocity $\mathbf{u}$ and write Eq. [3-30] as

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad [3\text{-}35]$$

where $\partial \mathbf{u}/\partial t$ is the time derivative in a fixed frame.

The Stress Tensor 3.3.2

When thermal motions are taken into account, a pressure force has to be added to the right-hand side of Eq. [3-35]. This force arises from the



Origin of the elements of the stress tensor. FIGURE 3-3

random motion of particles in and out of a fluid element and does not appear in the equation for a single particle. Let a fluid element $\Delta x\ \Delta y\ \Delta z$ be centered at $(x_0, \frac{1}{2}\Delta y, \frac{1}{2}\Delta z)$ (Fig. 3-3). For simplicity, we shall consider only the x component of motion through the faces A and B. The number of particles per second passing through the face A with velocity v_x is

$$\Delta n_v\, v_x\, \Delta y\, \Delta z$$

where Δn_v is the number of particles per m^3 with velocity v_x:

$$\Delta n_v = \Delta v_x \iint f(v_x, v_y, v_z)\, dv_y\, dv_z$$

Each particle carries a momentum mv_x. The density n and temperature KT in each cube is assumed to have the value associated with the cube's center. The momentum P_{A+} carried into the element at x_0 through A is then

$$P_{A+} = \Sigma\, \Delta n_v\, m v_x^2\, \Delta y\, \Delta z = \Delta y\, \Delta z [m\overline{v_x^2}\tfrac{1}{2}n]_{x_0-\Delta x} \qquad [3\text{-}36]$$

The sum over Δn_v results in the average $\overline{v_x^2}$ over the distribution. The factor $\frac{1}{2}$ comes from the fact that only half the particles in the cube at $x_0 - \Delta x$ are going *toward* face A. Similarly, the momentum carried out through face B is

$$P_{B+} = \Delta y\, \Delta z [m\overline{v_x^2}\tfrac{1}{2}n]_{x_0}$$

Thus the net gain in x momentum from right-moving particles is

$$\begin{aligned} P_{A+} - P_{B+} &= \Delta y\, \Delta z\, \tfrac{1}{2} m([n\overline{v_x^2}]_{x_0-\Delta x} - [n\overline{v_x^2}]_{x_0}) \\ &= \Delta y\, \Delta z\, \tfrac{1}{2} m(-\Delta x) \frac{\partial}{\partial x}(n\overline{v_x^2}) \end{aligned} \qquad [3\text{-}37]$$

This result will be just doubled by the contribution of left-moving particles, since they carry negative x momentum and also move in the opposite direction relative to the gradient of $n\overline{v_x^2}$. The total change of momentum of the fluid element at x_0 is therefore

$$\frac{\partial}{\partial t}(nmu_x)\, \Delta x\, \Delta y\, \Delta z = -m \frac{\partial}{\partial x}(n\overline{v_x^2})\, \Delta x\, \Delta y\, \Delta z \qquad [3\text{-}38]$$

Let the velocity v_x of a particle be decomposed into two parts,

$$v_x = u_x + v_{xr} \qquad u_x = \bar{v}_x$$

where u_x is the fluid velocity and v_{xr} is the random thermal velocity. For a one-dimensional Maxwellian distribution, we have from Eq. [1-7]

$$\tfrac{1}{2}m\overline{v_{xr}^2} = \tfrac{1}{2}KT \tag{3-39}$$

Equation [3-38] now becomes

$$\frac{\partial}{\partial t}(nmu_x) = -m\frac{\partial}{\partial x}[n(\overline{u_x^2} + 2\overline{uv_{xr}} + \overline{v_{xr}^2})] = -m\frac{\partial}{\partial x}\left[n\left(u_x^2 + \frac{KT}{m}\right)\right]$$

We can cancel two terms by partial differentiation:

$$mn\frac{\partial u_x}{\partial t} + mu_x\frac{\partial n}{\partial t} = -mu_x\frac{\partial(nu_x)}{\partial x} - mnu_x\frac{\partial u_x}{\partial x} - \frac{\partial}{\partial x}(nKT) \tag{3-40}$$

The equation of mass conservation*

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu_x) = 0 \tag{3-41}$$

allows us to cancel the terms nearest the equal sign in Eq. [3-40]. Defining the pressure

$$\boxed{p \equiv nKT} \tag{3-42}$$

we have finally

$$mn\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial p}{\partial x} \tag{3-43}$$

This is the usual pressure-gradient force. Adding the electromagnetic forces and generalizing to three dimensions, we have the fluid equation

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nabla p \tag{3-44}$$

What we have derived is only a special case: the transfer of x momentum by motion in the x direction; and we have assumed that the fluid is isotropic, so that the same result holds in the y and z directions. But it is also possible to transfer y momentum by motion in the x direction, for instance. Suppose, in Fig. 3-3, that u_y is zero in the cube at $x = x_0$ but is positive on both sides. Then as particles migrate across the faces A and B, they bring in more positive y momentum than they take out, and the fluid element gains momentum in the y direction. This *shear stress* cannot be represented by a scalar p but must be given by a tensor

* If the reader has not encountered this before, it is derived in Section 3.3.5.

P, the stress tensor, whose components $P_{ij} = mn\,\overline{v_i v_j}$ specify both the direction of motion and the component of momentum involved. In the general case the term $-\nabla p$ is replaced by $-\nabla \cdot \mathbf{P}$.

We shall not give the stress tensor here except for the two simplest cases. When the distribution function is an isotropic Maxwellian, **P** is written

$$\mathbf{P} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \qquad \text{[3-45]}$$

$\nabla \cdot \mathbf{P}$ is just ∇p. In Section 1.3, we noted that a plasma could have two temperatures $T_\perp$ and $T_\parallel$ in the presence of a magnetic field. In that case, there would be two pressures $p_\perp = nKT_\perp$ and $p_\parallel = nKT_\parallel$. The stress tensor is then

$$\mathbf{P} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix} \qquad \text{[3-46]}$$

where the coordinate of the third row or column is the direction of **B**. This is still diagonal and shows isotropy in a plane perpendicular to **B**.

In an ordinary fluid, the off-diagonal elements of **P** are usually associated with viscosity. When particles make collisions, they come off with an average velocity in the direction of the fluid velocity **u** at the point where they made their last collision. This momentum is transferred to another fluid element upon the next collision. This tends to equalize **u** at different points, and the resulting resistance to shear flow is what we intuitively think of as viscosity. The longer the mean free path, the farther momentum is carried, and the larger is the viscosity. In a plasma there is a similar effect which occurs even in the absence of collisions. The Larmor gyration of particles (particularly ions) brings them into different parts of the plasma and tends to equalize the fluid velocities there. The Larmor radius rather than the mean free path sets the scale of this kind of collisionless viscosity. It is a finite-Larmor-radius effect which occurs in addition to collisional viscosity and is closely related to the $\mathbf{v}_E$ drift in a nonuniform **E** field (Eq. [2-58]).

3.3.3 Collisions

If there is a neutral gas, the charged fluid will exchange momentum with it through collisions. The momentum lost per collision will be proportional to the relative velocity $\mathbf{u} - \mathbf{u}_0$, where $\mathbf{u}_0$ is the velocity of attributed at times to high-frequency oscillations. There has been no satisfactory resolution of the paradox, but this seems to be one of the few instances in plasma physics where nature works in our favor.

Another reason the fluid model works for plasmas is that the magnetic field, when there is one, can play the role of collisions in a certain sense. When a particle is accelerated, say by an **E** field, it would continuously increase in velocity if it were allowed to free-stream. When there are frequent collisions, the particle comes to a limiting velocity proportional to **E**. The electrons in a copper wire, for instance, drift together with a velocity $\mathbf{v} = \mu \mathbf{E}$, where μ is the mobility. A magnetic field also limits free-streaming by forcing particles to gyrate in Larmor orbits. The electrons in a plasma also drift together with a velocity proportional to **E**, namely, $\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2$. In this sense, a collisionless plasma behaves like a collisional fluid. Of course, particles do free-stream *along* the magnetic field, and the fluid picture is not particularly suitable for motions in that direction. *For motions perpendicular to* **B**, *the fluid theory is a good approximation.*

3.3.5 Equation of Continuity

The conservation of matter requires that the total number of particles N in a volume V can change only if there is a net flux of particles across the surface S bounding that volume. Since the particle flux density is $n\mathbf{u}$, we have, by the divergence theorem,

$$\frac{\partial N}{\partial t} = \int_V \frac{\partial n}{\partial t}\, dV = -\oint n\mathbf{u} \cdot d\mathbf{S} = -\int_V \nabla \cdot (n\mathbf{u})\, dV \qquad [3\text{-}49]$$

Since this must hold for any volume V, the integrands must be equal:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \qquad [3\text{-}50]$$

There is one such *equation of continuity* for each species. Any sources or sinks of particles are to be added to the right-hand side.

3.3.6 Equation of State

One more relation is needed to close the system of equations. For this, we can use the thermodynamic equation of state relating p to n:

$$p = C\rho^{\gamma} \qquad [3\text{-}51]$$