1.16 What is $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)$? Look at Griffiths p. 50

Is $\nabla\frac{1}{r}=\frac{-\hat{r}}{r^2}$ as in M3.6?

$$\nabla\frac{1}{r}=\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}\,\hat{x}+\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}\,\hat{y}+\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}\,\hat{z}$$

$$=-\tfrac{1}{2}(\;)^{-3/2}\,2x\,\hat{x}-\tfrac{1}{2}(\;)^{-3/2}\,2y\,\hat{y}-\tfrac{1}{2}(\;)^{-3/2}\,2z\,\hat{z}$$

$$=\frac{-(x\hat{x}+y\hat{y}+z\hat{z})}{(x^2+y^2+z^2)^{3/2}}=\frac{-\vec{r}}{r^3}=\frac{-\vec{r}}{r}\frac{1}{r^2}=\frac{-\hat{r}}{r^2}\checkmark$$

$$\nabla\frac{1}{r}=-\frac{\hat{r}}{r^2}$$

So $-\nabla\cdot\left(\nabla\frac{1}{r}\right)=\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)$

$$\nabla\cdot(\nabla T)=\nabla^2T=\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}+\frac{\partial^2T}{\partial z^2}$$

$$\nabla\cdot\left(\nabla\frac{1}{r}\right)=\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-1/2}+\frac{\partial^2}{\partial y^2}(x^2+y^2+z^2)^{-1/2}+\frac{\partial^2}{\partial z^2}(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-1/2}=\frac{\partial^2}{\partial x^2}(\;)^{-1/2}$$

$$=\frac{\partial}{\partial x}\left[-\tfrac{1}{2}(\;)^{-3/2}\,2x\right]=\frac{\partial}{\partial x}\left[-x(\;)^{-3/2}\right]$$

$$=-(\;)^{-3/2}+\tfrac{3}{2}x(\;)^{-5/2}\,2x=\frac{3x^2}{r^5}-\frac{1}{r^3}$$

$$\nabla\cdot\left(\nabla\frac{1}{r}\right)=\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right)+\left(\frac{3y^2}{r^5}-\frac{1}{r^3}\right)+\left(\frac{3z^2}{r^5}-\frac{1}{r^3}\right)$$

$$=\frac{3(x^2+y^2+z^2)}{r^5}-\frac{3}{r^3}=\frac{3r^2}{r^5}-\frac{3}{r^3}=\underline{0=-\nabla\cdot\left(\frac{\hat{r}}{r^2}\right)}$$

Check – spherical: $\nabla\cdot(\nabla f)=\nabla^2f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}f\right)$

$$\nabla\cdot\left(\nabla\frac{1}{r}\right)=\nabla^2\frac{1}{r}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\underbrace{\frac{\partial}{\partial r}\frac{1}{r}}_{-\frac{1}{r^2}}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-r^2}{r^2}\right)=0\checkmark$$

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x\hat{x} + y\hat{y} + z\hat{z}}{r \cdot r^2}\right)$$

$$\hat{x}\frac{\partial}{\partial x}\cdot\left(\frac{x\hat{x}}{r^3}\right) = \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial}{\partial x}x(x^2+y^2+z^2)^{-3/2}$$

$$= x\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial x}x$$

$$= x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) + (x^2+y^2+z^2)^{-3/2}$$

$$= \frac{-3x^2}{r^5} + \frac{1}{r^3}$$

Similarly, $\hat{y}\frac{\partial}{\partial y}\cdot\left(\frac{y\hat{y}}{r^3}\right) = -\frac{3y^2}{r^5} + \frac{1}{r^3}$

$\hat{z}\frac{\partial}{\partial z}\cdot\left(\frac{z\hat{z}}{r^3}\right) = -\frac{3z^2}{r^5} + \frac{1}{r^3}$

So $$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \left(\frac{-3x^2}{r^5} + \frac{1}{r^3}\right) + \left(-\frac{3y^2}{r^5} + \frac{1}{r^3}\right) + \left(-\frac{3z^2}{r^5} + \frac{1}{r^3}\right)$$

$$= -\frac{3(x^2+y^2+z^2)}{r^5} + \frac{3}{r^3} = \frac{-3r^2}{r^5} + \frac{3}{r^3} = \frac{-3}{r^3} + \frac{3}{r^3}$$

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 0$$

Griffiths p. 50 says $\nabla\cdot\frac{\hat{r}}{r^2} = 0$ except at origin

Arfken p. 81 : $\int \nabla\cdot\frac{\hat{r}}{r^2}\,d\tau = \begin{cases} 4\pi & \text{including origin} \\ 0 & \text{excluding origin}\end{cases}$