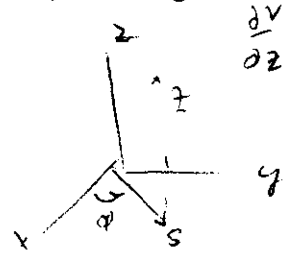
Separation of Variables: Laplace's equation $\nabla^2 V = 0$ in cylindrical coordinates

Worksheet for fall E&M Problem 3.23 (p.145)

Solve Laplace's eqn by separation of variables in cylindrical coordinates, assuming there is no dependence on z (this is cylindrical symmetry).

$$\frac{\partial V}{\partial z} = \frac{\partial S(s)\Phi(\varphi)}{\partial z} = 0$$



The Laplacian in cylindrical coordinates is eqn (1.82) p.44:

(1) $$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

Look for solutions of the form $V(s,\phi) = S(s)\,\Phi(\phi)$ (2)

$$\frac{\partial V}{\partial s} = \Phi\frac{\partial S}{\partial s}, \quad \nabla^2 V = \Phi\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{s^2}S\frac{\partial^2\Phi}{\partial\varphi^2} + 0$$

Multiply by s^2 and divide by $V = S\,\Phi$:

$$\frac{\Phi}{S\Phi}\frac{s^2}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{s^2}{s^2}\frac{S}{S\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = \frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

Both terms must be constant, and they must sum to zero, so the two constants are equal and opposite. Choose $-k^2$ for the Φ solution so it returns to its original value in one cycle.

Find solutions to (3) and (4)

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = k^2 = C_1 \qquad\qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = C_2 = -k^2$$

(4) Since $\Phi(\phi + 2\pi) = \Phi(\phi)$, k must be an integer.

$$\frac{d^2\Phi}{d\varphi^2} = -k^2\Phi \longrightarrow \Phi = \Phi_0 e^{ik\varphi}$$

Show that $S = s^n$ is a solution to (3): what is the relation between n and k?

$$\frac{\partial S}{\partial s} = \frac{\partial}{\partial s}s^n = ns^{n-1}, \quad \frac{d}{ds}\left(s\left[ns^{n-1}\right]\right) = n\frac{\partial}{\partial s}s^n = n^2 s^{n-1}, \quad \frac{s}{S}\left[n^2 s^{n-1}\right] = k^2$$

$$\frac{1}{s^n}n^2 s^n = k^2$$

$$\boxed{n = \pm k}$$

Show that for k not 0, $S = A\,s^k + B\,s^{-k}$, and for $\boxed{k=0}$ $S = D + C \ln s$.

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0$$

$$\left(s\frac{dS}{ds}\right) = \text{const} \rightarrow \int dS = S = \int C\frac{ds}{s} = \underline{C\ln s + D}$$

for $k = 0$ ✓

$$\underline{S = s^n = As^k + Bs^{-k}}$$ ✓

for $k \neq 0$

What is Φ for the k=0 case? Put it all together into a general solution. Then apply it to problem 3.24.

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = 0 \quad ; \quad \frac{d\Phi}{d\varphi} = \text{constant} = a$$

no good; not periodic in φ.

$$\Phi = \int d\Phi = \int a\,d\varphi = a\varphi + b \longrightarrow \underline{\Phi = b} \quad k = 0$$

SOLUTION: $$V = S\Phi = \underbrace{C\ln s + D}_{k=0} + \sum_{k=1}^{\infty}\left(As^k + Bs^{-k}\right)e^{\pm ik\varphi}$$

fall EM HW #3 Continued two week! Physical Systems Zita - 1 Nov 02

EM wk 4-5 HW due wk 6 Ch 3 #5, 9, 12, 13, 16 (was #17 in Ed 3)

Also work through Ex 3.3 and Ex 6.1

See work sheet for #12, & #23

p. 136 #13: For the infinite slot in Ex 3.3, find the charge density σ on the strip at $x=0$, assuming it is a conductor at constant V_0.

~~We~~ Griffiths found $V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} e^{-n\pi x/a} \sin \frac{n\pi y}{a}$

Recall from Ch 2 that across a charge distribution σ, the E field changes: $\Delta E = \frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial n}$ where n is the direction across the charge. Here, $\frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial x}$

$$\frac{\partial V}{\partial x} = \frac{4V_0}{\pi} \sum_{\text{odd } n} \frac{1}{n} \left(\frac{\partial}{\partial x} e^{-n\pi x/a} \sin \frac{n\pi y}{a} \right)$$

$$\left(e^{-n\pi x/a} \frac{\partial}{\partial x} \sin \frac{n\pi y}{a} + \sin \frac{n\pi y}{a} \frac{\partial}{\partial x} e^{-\frac{n\pi x}{a}} \right)$$

$$\left(e^{-n\pi x/a} \cdot 0 + \sin \frac{n\pi y}{a} \left(-\frac{n\pi}{a} \right) e^{-\frac{n\pi x}{a}} \right)$$

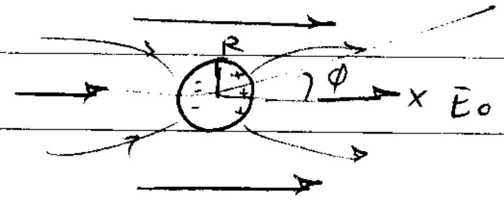
$$\frac{\partial V}{\partial x} = \frac{4V_0}{\pi} \sum \frac{1}{n} \left(\frac{-n\pi}{a} \right) e^{-n\pi x/a} \sin \frac{n\pi y}{a}$$

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = \frac{-4V_0}{a} \sum \sin \frac{n\pi y}{a} = -\frac{\sigma}{\epsilon_0}$$

$$\boxed{\sigma = +\frac{4V_0 \epsilon}{a} \sum_{\text{odd } n} \sin \frac{n\pi y}{a}}$$

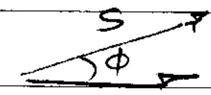
EM WORKSHEET

3.24: Use our results from #23 to find the potential
145 outside an infinitely long pipe of radius R,
placed at right angles to an otherwise uniform E_0.
Then find σ on pipe.



Induced charge separation on pipe
(i) perturbs E near pipe. Let $V(R)=0$: pipe is grounded.
If we can find $V(s>R)$, then we can find $\vec{E}=-\vec{\nabla}V$
(ii) BC: $\vec{E}=E_0\hat{x}$ for $s \gg R$

$$-\frac{\partial V}{\partial x}=E_0 \rightarrow V(s\gg R)=-E_0 x$$



$$V(s\gg R)=-E_0 s\cos\phi \qquad x=s\cos\phi$$

(from #23) Apply these BC to $V(s,\phi)=C\ln s + D + \sum_{k=1}^{\infty}(As^k+Bs^{-k})e^{\pm ik\phi}$

$$0=V(R,\phi)=C\ln R + D + \sum(AR^k+BR^{-k})e^{\pm ik\phi}$$

these must vanish — write this as sin & cos:

$$V(s,\phi)=\sum_{k=1}^{\infty}\left[s^k(a\cos k\phi+b\sin k\phi)+s^{-k}(c\cos k\phi+d\sin k\phi)\right]$$

there are no sine terms, so
$b=d=0$

There are ϕ terms, but only for $k=0$

$V(s,\phi)=$

Then apply BC(i) to relate a to c: