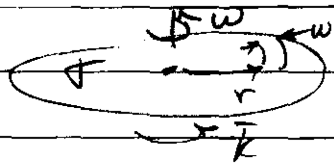
E&M HW #5 - Physical Systems - due 27 Feb 2007

Ch 5 # 6, 12, 13, 16 (XC: 17) E/Z

Problem 5.6

(a) A phonograph record carries a uniform density of "static electricity" σ. If it rotates at angular velocity ω, what is the surface current density K at a distance r from the center?

(b) A uniformly charged solid sphere, of radius R and total charge Q, is centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\mathbf{J}$ at any point (r, θ, ϕ) within the sphere.



width perpendicular to flow

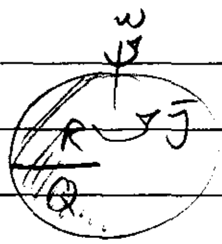
Surface current density $\bar{K} = \dfrac{\text{current}}{\text{width}_\perp} = \sigma \bar{v}$ (5.23) 2.11

The speed v depends on the distance r from the center:

$s = r\theta$

$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \rightarrow \vec{K} = \sigma\vec{v} = \sigma r \omega \hat{\theta}$

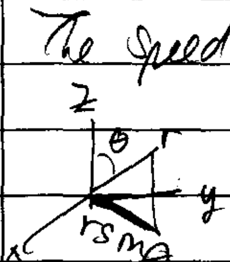
(b)



Volume current density $\bar{J} = \dfrac{dI}{da_\perp} = \dfrac{\text{current}}{\text{area}_\perp} = \rho \bar{v}$

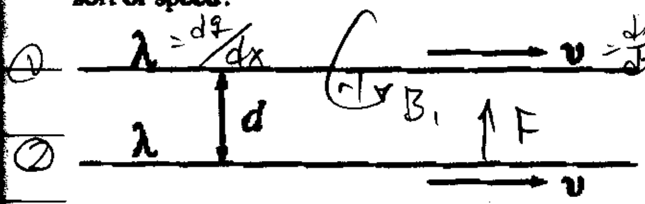
The charge density $\rho = \dfrac{\text{charge}}{\text{volume}} = \dfrac{Q}{\frac{4}{3}\pi R^3}$

The speed $v = r'\omega$ where $r' = r\sin\theta$ = radial distance projected perpendicular to z-axis



So $\vec{J} = \rho v r\sin\theta = \dfrac{Q v r \sin\theta}{\frac{4}{3}\pi R^3} \hat{\phi}$

Problem 5.12 Suppose you have two infinite straight line charges λ, a distance d apart, moving along at a constant speed v (Fig. 5.26). How great would v have to be in order for the magnetic attraction to balance the electrical repulsion? Work out the actual number... Is this a reasonable sort of speed?[8]



current $= \frac{dq}{dt} = \frac{dq}{dx}\frac{dx}{dt}$

$I = \lambda v$

$$\oint B_1 \cdot dl = \mu_0 I_1$$

$$B_1 \cdot 2\pi r = \mu_0 \lambda v \rightarrow B_1(r) = \frac{\mu_0 \lambda v}{2\pi r}$$ due to top line.

Force on bottom line carrying $I_2 = \lambda v$ due to top line is $F_B = I_2 l \times B_1$ (UP)

Force per unit length is $\frac{F_B}{l} = |I_2 \times B_1| = \lambda v \left(\frac{\mu_0 \lambda v}{2\pi d}\right)$

$$\frac{F_B}{l} = \frac{\mu_0}{2\pi d}(\lambda v)^2$$

Electrical repulsion on bottom line (DOWN) is due to electric field of top line, given by (2.9) p. 63

$$E = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s} = \frac{\lambda}{2\pi\epsilon_0 d}$$

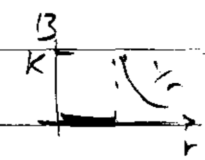
$$\frac{F_E}{l} = \frac{q}{l}E = \lambda E = \frac{\lambda^2}{2\pi\epsilon_0 d}$$

$$\frac{F_E}{l} = \frac{F_B}{l} \text{ if } \frac{\mu_0}{2\pi d}(\lambda v^2) = \frac{\lambda^2}{2\pi\epsilon_0 d} \rightarrow v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$

Problem 5.13 A steady current I flows down a long cylindrical wire of radius a (Fig. 5.40). Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that J is proportional to s, the distance from the axis.



[redacted]

(a) $I(r \geq a) = 2\pi a k$, $I(r<0) = 0$

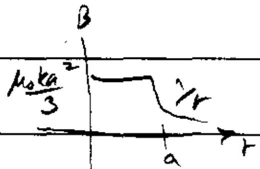
$$\oint \vec{B} \cdot d\vec{l} = B(r)\, 2\pi r = 2\pi a k$$

$$B(r>a) = \frac{ak}{r}, \quad B(r<0) = 0$$

(b) We showed in 5.5 that for $J = kr$, $I(r) = \frac{2\pi k}{3} r^3$

$$\oint \bar{B} \cdot d\bar{l} = 2\pi r B(r) = \mu_0 I(r) \qquad I_{tot} = \frac{2\pi k}{3} a^3$$

$$= \mu_0 \frac{2\pi k}{3} r^3$$



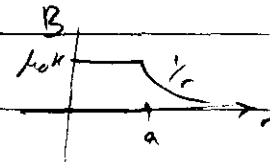
$$B(r<a) = \frac{\mu_0 k r^3}{3r} = \frac{\mu_0 k r^2}{3}, \quad B(a) = \frac{\mu_0 k a^2}{3}$$

$$B(r>a) = \frac{\mu_0}{2\pi r} I_{total} = \frac{\mu_0}{2\pi r} \frac{2\pi k}{3} a^3 = \frac{\mu_0 k a^3}{3r}$$

(c) In 5.5 (b), we had $J = \frac{k}{r} \rightarrow I(r) = 2\pi r k$

Then inside, $\oint B \cdot dl = 2\pi r B(r) = \mu_0 2\pi r k$

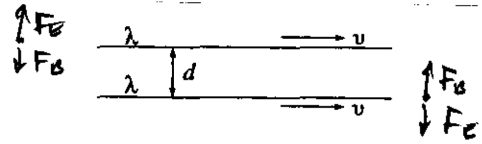
$$B(r<a) = \mu_0 k$$



Outside, $I_{tot} = 2\pi a k$, $B(r>a) = \mu_0 k \frac{a}{r}$

Problem 5.12 Suppose you have two infinite straight line charges λ, a distance d apart, moving along at a constant speed v (Fig. 5.26). How great would v have to be in order for the magnetic attraction to balance the electrical repulsion? Work out the actual number... Is this a reasonable sort of speed?[8]

[8] If you've studied special relativity, you may be tempted to look for complexities in this problem that are not really there—λ and v are both measured *in the laboratory frame*, and this is *ordinary electrostatics* (see footnote 4).

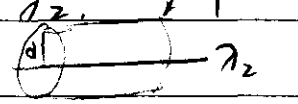


$\lambda = \frac{\text{charge}}{\text{length}} = \frac{q}{l} \rightarrow q = \lambda l$

$I = \lambda v$

Consider the electrostatic repulsion of λ_1 by the field E_2 of λ_2:

(1) $F = q_1 E_2 \rightarrow \frac{F}{l} = \lambda_1 E_2$ where $\int E_2 \cdot da = \frac{q}{\epsilon_0}$: find the field $E_2(d)$ a distance d away from the source λ_2. Gaussian cylinder



$\int da$ = area of cylinder $= 2\pi d \cdot l$

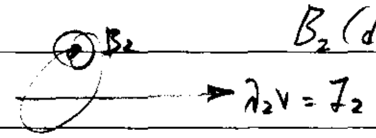
(2) $$E_2(d) = \frac{q/\epsilon_0}{\text{area}} = \frac{\lambda l/\epsilon_0}{2\pi d\, l} = \frac{\lambda_2}{2\pi\epsilon_0 d}$$

Substitute (2) into (1) to find the mutual electrostatic repulsion of the two lines of charge: $\frac{F_E}{l} = \lambda_1 E_2 = \frac{\lambda_1 \lambda_2}{2\pi\epsilon_0 d} = \frac{\lambda^2}{2\pi\epsilon_0 d}$

Now consider the magnetostatic attraction of I_1 by the field B_2:

current $\frac{dq}{dt} = \frac{dq}{dx}\frac{dx}{dt} = \lambda v$, First find the field due to I_2:

(3) $\oint B_2 \cdot dl = \mu_0 I_2 = \mu_0 \lambda_2 v$



$$B_2(d) = \frac{\mu_0 \lambda_2 v}{2\pi d}$$

Then find the force between the two currents, in terms of v, λ, and d:

(4) $$\frac{|F_M|}{l} = |\bar{I}_1 \times \bar{B}_2| = \lambda v\left(\frac{\mu_0 \lambda v}{2\pi d}\right) = \frac{\mu_0}{2\pi d}\lambda^2 v^2$$

Finally, consider what is required for these two forces to balance:

(2) $$\frac{F_E}{l} = \frac{F_M}{l} \qquad (4)$$

$$\frac{\lambda^2}{2\pi\epsilon_0 d} = \frac{\mu_0}{2\pi d}\lambda^2 v^2$$

$$1 = \mu_0\epsilon_0 v^2$$

At what speed v does this relationship hold?

$$v = \sqrt{\frac{1}{\mu_0\epsilon_0}} = \sqrt{\frac{1}{4\pi\times10^{-7}\,N\left(\frac{s}{C}\right)^2\,8.85\times10^{-12}\,\frac{C^2}{Nm^2}}}$$

$$v = 3\times10^8\,\frac{m}{s^2} = \text{speed of light!}$$

Since this speed is unattainable, $F_M < F_E$ and repulsion wins over attraction.

(electrostatic) permittivity of free space $\epsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{Nm^2}$

(magnetostatic) permeability of free space $\mu_0 = 4\pi\times10^{-7}\left[\frac{Tm}{A} = \frac{N}{Am}\frac{m}{A} = \frac{N}{A^2}\right]$

$$B = \frac{F}{qv} \;:\; [T] = \frac{[N]}{[C\cdot m/s]} = \frac{N}{m}\,\frac{s}{C} = \frac{N}{Am}$$

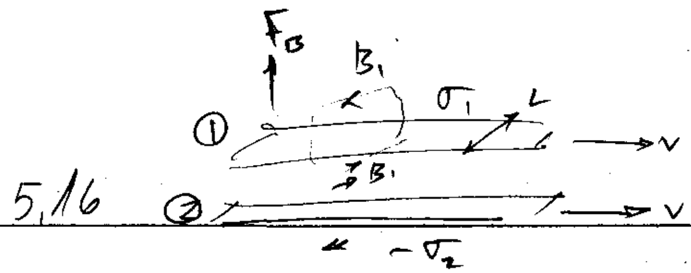
Problem 5.16 A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Fig. 5.43.

(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed v would the magnetic force balance the electrical force?[11]

5.16



ⓐ $\oint B \cdot dl = \mu_0 I \qquad I = \frac{dq}{dt} = \frac{dq}{da}\frac{da}{dx}\frac{dx}{dt}$

$I = \sigma L V$

$2BL = \mu_0 \sigma L V$

$B_1 = \frac{\mu_0 \sigma V}{2} = B_2$

B between plates $= B_1 + B_2 = \mu_0 \sigma V$

ⓑ $\frac{F_B}{A} = \frac{I \bar{l} \times B}{A} = \frac{dq/dt \; l \times B}{A} = \sigma \vec{v} \times \vec{B}_2$

$= \sigma v \left(\mu_0 \sigma \frac{V}{2}\right)$

$\frac{F_B}{A} = \frac{\mu_0}{2}(\sigma v)^2$

ⓒ Electrical force of attraction on plate 1 is due to the electric field due to plate 2

From Ex 2.5 (p.74), $E = \frac{\sigma}{\epsilon_0}$ inside, $E_2 = \frac{\sigma}{2\epsilon_0}$

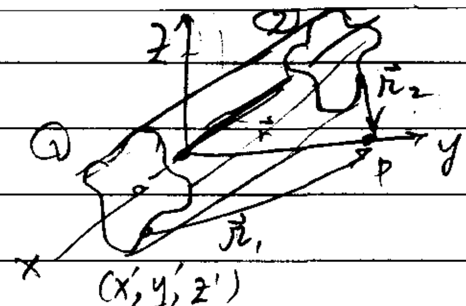
$\frac{F_E}{A} = \frac{q}{A} E = \sigma E = \frac{\sigma^2}{2\epsilon_0}$

If $\frac{F_E}{A} = \frac{F_B}{A}$ then $\frac{\sigma^2}{2\epsilon_0} = \frac{\mu_0}{2}(\sigma v)^2 \rightarrow v = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$

$v = c$

(a)

Problem 5.17 Show that the magnetic field of an infinite solenoid runs parallel to the axis, *regardless of the cross-sectional shape of the coil*, as long as that shape is constant along the length of the solenoid. What is the magnitude of the field, inside and outside of such a coil? (b) (c) Show that the toroid field (5.58) reduces to the solenoid field, when the radius of the donut is so large that a segment can be considered essentially straight.



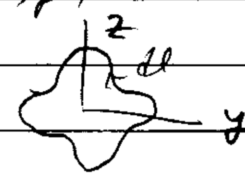
Consider a solenoid with a cross-sectional shape as in Fig $\frac{5.52}{230}$ aligned along the x-axis. Find the field at point P on the y axis: $\vec{r} = (0, y, 0)$

Consider a source point on loop (1) at (x', y', z'):

$\vec{r}_1 = -x'\hat{x} + (y-y')\hat{y} + z'\hat{z}$

$d\vec{l}' = dy'\hat{y} + dz'\hat{z}$

(The current has y and z components)



For a source point at $(-x', y', -z')$ on loop (2):

$\vec{r}_2 = +x'\hat{x} + (y-y')\hat{y} + z'\hat{z}$

For each loop, the field at $\vec{r}$ due to the current element at r' is given by the Biot-Savart law

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{I} \times \vec{r}\, dl'}{r^3} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l}' \times \vec{r}}{r^3}$$

Loop (1)

$$d\vec{l}' \times \vec{r}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & dy' & dz' \\ -x' & (y-y') & z' \end{vmatrix} = \begin{matrix} \hat{x}\,(z'dy' - (y-y')dz') \\ -\hat{y}\,(0 + x'dz') \\ +\hat{z}\,(0 + x'dy') \end{matrix}$$

and $r_1^{\,3} = [(x')^2 + (y-y')^2 + (z')^2]^{3/2}$.

Loop ②: $d\vec{l}' \times \vec{r}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & dy' & dz' \\ x' & (y-y') & z' \end{vmatrix} = \begin{matrix} \hat{x}\left(z'dy' - (y-y')dz'\right) \\ -\hat{y}\left(0 - x'dz'\right) \\ +\hat{z}\left(0 - x'dy'\right) \end{matrix}$

$$r_2^3 = \left[(x')^2 + (y-y')^2 + (z')^2\right]^{3/2} = r_1^3$$

The contribution of loop ① to B ($d\vec{l}' \times \vec{r}_1$) is the same as " " ② ($d\vec{l}' \times \vec{r}_2$) except that the $\hat{y}$ and $\hat{z}$ terms CANCEL (because x changed sign).

Therefore only the $\hat{x}$ component survives: $\vec{B} = B_x \hat{x}$

This proves that the field is along the axis regardless of (constant) coil shape (QED).

ⓑ Therefore, we can simply use Ampere's law (which is much easier!) ($B_{out} = 0$)

$$\oint B \cdot dl = \mu_0 N I = B \cdot L \rightarrow \vec{B}_{inside} = \frac{\mu_0 N I}{L} \hat{x} = \mu_0 n I$$

where N = the number of turns

or $n = N/L$ = number of turns per unit length

ⓒ Toroid field (5.58) 230 $\vec{B}(r) = \frac{\mu_0 N I}{2\pi s} \hat{\phi}$ $\qquad \frac{N}{2\pi s} = n$

$$\vec{B}(r)_{INSIDE} = \mu_0 n I \hat{\phi} \text{ (Same)}$$

($B_{out} = 0$)