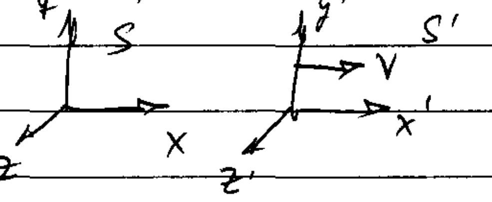
Modern Physics Ch 1 HW due 25 Jan 2007

E1 Zta

Derive γ, #8, 44, 49-50

Derive γ for (relativistic) Lorentz transformation between x-coordinate in rest frame S and x' coordinate in frame S' moving with speed v in x direction



Galilean : $x' = x - vt$

(1-13) Lorentz : $x' = \gamma(x - vt)$

(1-14) Inverse : $x = \gamma(x' + vt')$

First, solve (1-13) and (1-14) for t' by eliminating x':

$$x = \gamma\left[\gamma(x - vt) + vt'\right] = \gamma^2(x - vt) + \gamma vt'$$

$$\gamma vt' = -\gamma^2(x - vt) + x$$

$$t' = -\frac{\gamma(x - vt)}{v} + \frac{x}{\gamma v} = \gamma\left[t + \frac{x}{v}\left(\frac{1}{\gamma^2} - 1\right)\right] =$$

(1-15) $$t' = \gamma\left[t + \frac{x}{v}\left(\frac{1 - \gamma^2}{\gamma^2}\right)\right] \checkmark$$

This transformation must work for a flash of light at the origin in either the

(1-16) rest frame $x^2 + y^2 + z^2 = c^2 t^2$ or the

(1-17) moving frame $x'^2 + y'^2 + z'^2 = c^2 t'^2$. (note $y' = y$, $z' = z$)

Substitute (1-13) and (1-15) into (1-17):

$$\gamma^2(x - vt)^2 + \underbrace{y'^2}_{=y^2} + \underbrace{z'^2}_{=z^2} = c^2\gamma^2\left[t + \frac{x}{v}\left(\frac{1 - \gamma^2}{\gamma^2}\right)\right]^2$$

(1) $\gamma^2(x^2-2xvt+v^2t^2)+y^2+z^2 = c^2\gamma^2\left[t^2+\frac{x^2}{v^2}\left(\frac{1-\gamma^2}{\gamma^2}\right)^2 + 2t\frac{x}{v}\left(\frac{1-\gamma^2}{\gamma^2}\right)\right]$

(2) This must ~~transform~~ to (1-16) $x^2+y^2+z^2=c^2t^2$

Let's just match the x^2 terms in (1) and (2)

(1) (2)

$$x^2 = x^2\gamma^2 - c^2\gamma^2\frac{x^2}{v^2}\left(\frac{1-\gamma^2}{\gamma^2}\right)^2$$

$$1 = \gamma^2\left[1-\frac{c^2}{v^2}\left(\frac{1-\gamma^2}{\gamma^2}\right)^2\right] = \gamma^2\left[1+\frac{1}{\beta^2}\left(\frac{1+\gamma^2}{\gamma^2}\right)^2\right]$$

or, better

$$1-\gamma^2 = -\frac{c^2}{v^2}\frac{(1-\gamma^2)^2}{\gamma^2} = -\frac{1}{\beta^2}\frac{(1-\gamma^2)^2}{\gamma^2}$$

$$1 = \frac{-(1-\gamma^2)}{\beta^2\gamma^2}$$

$$\beta^2\gamma^2 = \gamma^2-1$$

$$1 = \gamma^2(1-\beta^2) \rightarrow \gamma^2 = \frac{1}{1-\beta^2}$$

$$\boxed{\gamma = \sqrt{\frac{1}{1-\beta^2}} = \sqrt{\frac{1}{1-v^2/c^2}}}$$

(1-8.) Consider two inertial reference frames. When an observer in each frame measures the following quantities, which measurements made by the two observers *must* yield the same results? Explain your reason for each answer.

(*a*) The distance between two events No - length contraction

(*b*) The value of the mass of a proton No - apparent or measured mass depends on relative motion

(*c*) The speed of light yes - fundamental postulate

(*d*) The time interval between two events no - time dilation

(*e*) Newton's first law no - $F = \gamma \, dp/dt$

(*f*) The order of the elements in the periodic table yes - number of nucleons is unchanged

(*g*) The value of the electron charge yes - charge is invariant, though fields are not

1-44. H. A. Lorentz suggested 15 years before Einstein's 1905 paper that the null effect of the Michelson-Morley experiment could be accounted for by a contraction of that arm of the interferometer lying parallel to Earth's motion through the ether to a length $L = L_p(1 - v^2/c^2)^{-1/2}$. He thought of this, incorrectly, as an actual shrinking of matter. By about how many atomic diameters would the material in the parallel arm of the interferometer have had to shrink in order to account for the absence of the expected shift of 0.4 of a fringe width? (Assume the diameter of atoms to be about 10^{-10} m.)

L_p = proper Length

$L = \frac{L_p}{\gamma}$ = contracted

$$\Delta L = L_p - L = L_p - \frac{L_p}{\gamma} = L_p\left(1 - \frac{1}{\gamma}\right)$$

Binomial expansion: $\frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$

$$= 1 + \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2!}\left(-\frac{v^2}{c^2}\right)^2 + \ldots$$

$$= 1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{1}{8}\frac{v^4}{c^4} + \ldots$$

Since $v_{earth} \ll c$, approximate $\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$

then $\left(1 - \frac{1}{\gamma}\right) \approx 1 - \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2}v^2/c^2$

p.12 Length of interferometer arms L_p = 11 m

p.9 Speed of Earth "through ether" $v = 3 \times 10^4$ m/s, $\frac{v}{c} = \frac{3 \times 10^4}{3 \times 10^8} = 10^{-4}$

$$\left(1 - \frac{1}{\gamma}\right) = \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}10^{-8}$$

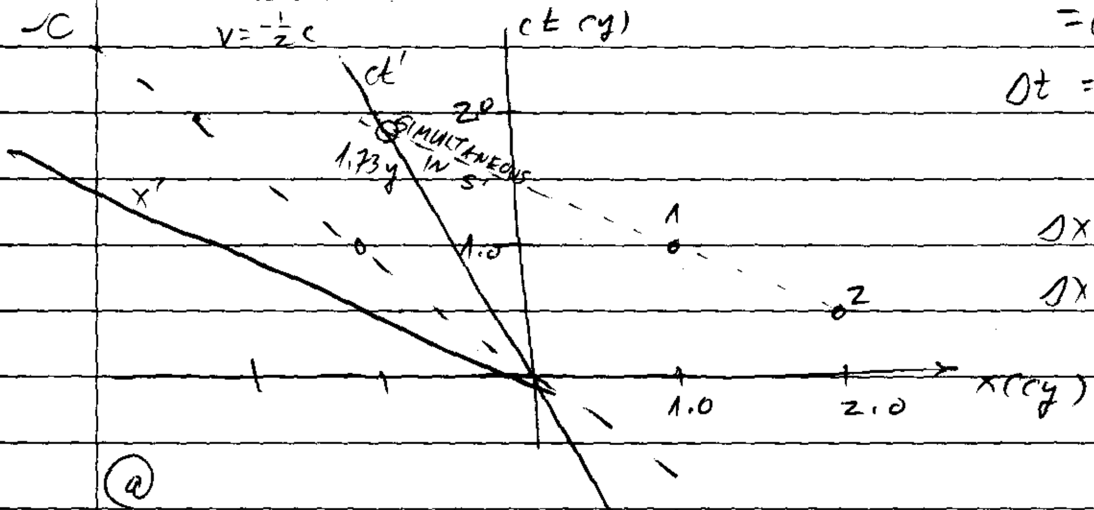
$$\Delta L = L_p\left(1 - \frac{1}{\gamma}\right) = 11\text{ m}\left(\frac{1}{2}10^{-8}\right) = 5.5 \times 10^{-8}\text{ m}$$

"Shrinkage" $= \frac{\Delta L}{10^{-10}\text{ m/atomic diameter}} = 5.5 \times 10^{-8} = 5.5 \times 10^2 = 550$ atomic diam

1-49. Frames S and S' are moving relative to each other along the x and x' axes. They set their clocks to $t = t' = 0$ when their origins coincide. In frame S, event 1 occurs at $x_1 = 1\ c \cdot y$ and $t_1 = 1$ y and event 2 occurs at $x_2 = 2.0\ c \cdot y$ and $t_2 = 0.5$ y. These events occur simultaneously in frame S'. (*a*) Find the magnitude and direction of the velocity of S' relative to S. (*b*) At what time do both of these events occur as measured in S'? (*c*) Compute the spacetime interval Δs between the events. (*d*) Is the interval spacelike, timelike, or lightlike? (*e*) What is the proper distance L_p between the events?

1-50. Do Problem 1-49 parts (*a*) and (*b*) using a spacetime diagram.

$\Delta t = t_2 - t_1 =$
$= 0.5 - 1\ y$
$\Delta t = -0.5\ y$



$\Delta x = 2 - 1\ cy$
$\Delta x = 1\ cy$

ⓐ

(1-22) 23 Time dilation $\Delta t' = \gamma \Delta t - \frac{\gamma v}{c^2} \Delta x$

Simultaneous in S': $\Delta t' = 0 \rightarrow \Delta t' = 0 \rightarrow \gamma \Delta t = \frac{\gamma v}{c^2} \Delta x$

Solve for $v = \frac{\Delta t\, c^2}{\Delta x} = \frac{(-0.5\ y) \cdot c^2}{1\ cy} = -0.5\ c$ = motion of frame S'

ⓑ At what time do these events occur as measured in S'?

(1-20) 23 $t' = \gamma \left(t - \frac{vx}{c^2}\right)$ $\quad \gamma = \frac{1}{\sqrt{1 - 1/4}} = \frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}}$

$= 1.73\ y$

$t' = \frac{2}{\sqrt{3}}\left(1_y - \frac{(-\frac{1}{2}c)\, 1\, c\, y}{c^2}\right) = \frac{2}{\sqrt{3}}\left(1 + \frac{1}{2}\ y\right) = \frac{2}{\sqrt{3}}\left(\frac{3}{2}\ y\right) = \sqrt{3}\ y$

This was for $(x, t) = (1, 1)$. Same for $(x, t) = (2, \frac{1}{2})$

ⓒ Spacetime interval $\Delta S = \sqrt{\Delta x^2 - (c\Delta t)^2} = \sqrt{1^2 - \frac{1}{2}^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = 0.87\ cy$

ⓓ $(\Delta x = 1) > (\Delta ct = \frac{1}{2})$ so the interval is SPACELIKE. not causally connected events

ⓔ $L_p = \Delta S = 0.87\ c \cdot yr$

Modern Physics Ch 2 - Relativity II #13, 35, 48

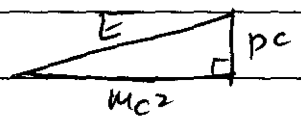
13. The total energy of a particle is twice its rest energy. (*a*) Find *u/c* for the particle. (*b*) Show that its momentum is given by $p = (3)^{1/2} mc$.

$$E = \gamma mc^2 \rightarrow \gamma = \frac{E}{mc^2} = 2$$

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \rightarrow \text{find } \frac{u}{c} = \sqrt{1-\frac{1}{\gamma^2}} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}}$$

$$\frac{1}{\gamma^2} = 1 - \frac{u^2}{c^2} \rightarrow \frac{u}{c} = \frac{\sqrt{3}}{2} = 0.87$$

(b)



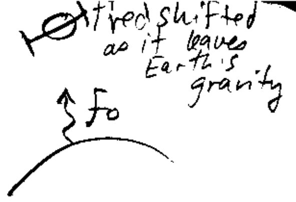
$$(pc)^2 + (mc^2)^2 = E^2$$

$$(pc)^2 = E^2 - (mc^2)^2 = (2mc^2)^2 - (mc^2)^2$$

$$= (4\,mc^2)^2 - (mc^2)^2 = 3(mc^2)^2$$

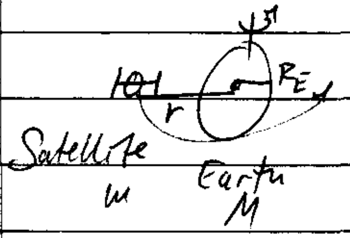
$$pc = \sqrt{3}\, mc^2$$

(GPS satellites must make similar corrections)



2-35. A synchronous satellite "parked" in orbit over the equator is used to relay microwave transmissions between stations on the ground. To what frequency must the satellite's receiver be tuned if the frequency of the transmission from Earth is exactly $f_0 =$ 9.375 GHz? (Ignore all Doppler effects.)

Find the radius and g for geosynchronous orbits : $T = 1$ day



① $F = \frac{GmM}{r^2} = \frac{mv^2}{r} \rightarrow v^2 = \frac{GM}{r}$

② $v^2 = \left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$

Eliminate v from ① & ②, solve for r = satellite orbit radius

$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$

$r^3 = \frac{GMT^2}{4\pi^2} = \frac{6.67\times10^{-11}\frac{N\,m^2}{kg^2}\cdot 6\times10^{24}\,kg \times \left(1\,day \left|\frac{24\,hr}{d}\right|\frac{3600\,s}{hr}\right)^2}{4\pi^2}\left|\frac{kg\,m/s^2}{N}\right.$

$r = 4.23\times10^7$ m

Then you can find g(r) at the satellite orbit radius from $\frac{GmM}{r^2} = mg$

where M = mass of Earth, G = grav. constant

$g = \frac{GM}{r^2} = \frac{6.67\times10^{-11}\frac{kg\,m^3}{kg^2\,s^2}\cdot 6\times10^{24}\,kg}{(4.23\times10^7\ m)^2} = 2.24\times10^{-1}\ \frac{m}{s^2}$

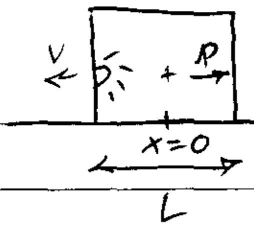
Once you have g(r), you can use the general relativistic frequency shift $\Delta f = f_0 - f' = \frac{f_0 g h}{c^2}$ (2-45)

Where h = altitude = $r - R_{Earth} = 4.23\times10^7\ m - 6.37\times10^6\ m = 3.59\times10^7\ m$

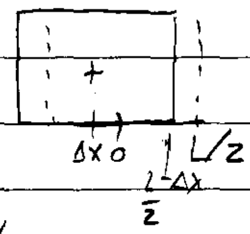
$\Delta f = 9.375\ GHz\ \frac{\left(0.224\ \frac{m}{s^2}\right)\left(3.59\times10^7\ m\right)}{\left(3\times10^8\ \frac{m}{s}\right)^2} = 8.36\times10^{-10}$ GHz

$f' = f_0 - \Delta f$: The frequency is LOWER by a part in a BILLION

2-48. In a simple thought experiment, Einstein showed that there is mass associated with electromagnetic radiation. Consider a box of length L and mass M resting on a frictionless surface. At the left wall of the box is a light source that emits radiation of energy E, which is absorbed at the right wall of the box. According to classical electromagnetic theory, this radiation carries momentum of magnitude $p = E/c$. (*a*) Find the recoil velocity of the box such that momentum is conserved when the light is emitted. (Since p is small and M is large, you may use classical mechanics.) (*b*) When the light is absorbed at the right wall of the box, the box stops, so the total *momentum* remains zero. If we neglect the very small velocity of the box, the time it takes for the radiation to travel across the box is $\Delta t = L/c$. Find the distance moved by the box in this time. (*c*) Show that if the center of mass of the system is to remain at the same place, the radiation must carry mass $m = E/c^2$.



(a) $P_{light} = \frac{E}{c} = -(P_{box} = MV) \to V = \frac{E}{Mc}$ = recoil speed of box



(b) $\Delta X = V\Delta t$, $\Delta t = \frac{L}{c}$

$\Delta X = \frac{L}{Mc}\frac{L}{c} = \frac{EL}{Mc^2}$ = distance box moves before light is absorbed

(c) Say radiation of mass m is emitted from the left side of the box at $x = -\frac{L}{2}$. Then the center of mass of the box is at (originally)

$$x_{cm_1} = \frac{\sum m_i x_i}{\sum m_i} = \frac{M\cdot 0 + m(\frac{L}{2})}{M+m} =$$

$$(M+m)\,x_{cm_1} = -m\frac{L}{2}$$

When the radiation is absorbed on the right side of the box, the box has moved a bit, and the center of mass is now at:

$$(M+m)X_{cm_2} = M(-\Delta X) + m\left(\frac{L}{2} - \Delta X\right) = -m\frac{L}{2} = (M+m)x_{cm_1}$$

$$m\frac{L}{2} + m\frac{L}{2} = m\Delta X + M\Delta X = mL$$

$$M\Delta x = m(L - \Delta X)$$

Since this is an internal process, the center of mass cannot move: $x_{cm_1} = x_{cm_2}$. Solve for m:

$$M\Delta X = m\left(L - \frac{EL}{Mc^2}\right)$$

$$M\frac{EL}{Mc^2} = mL\left(1 - \frac{E}{Mc^2}\right) \to \frac{E}{c^2} \approx m(1-0) \checkmark$$

→ 0 since $E \ll Mc^2$