Modern Physics HW #4 due 22 Feb 2007 — 5/7

Modern physics Ch.6 HW a: 7, 19, 25, 27, and from lecture:
Exercise in probability, infinite square well centered on x=0, and show that stationary states are separable.

Exercises in probability: quantitative

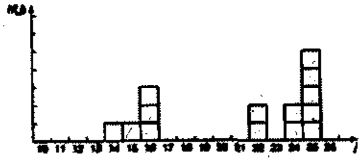


Figure 1.6: Histogram showing the number of people, $N(j)$, with age j, for the example in Section 1.3.

1. Probability that an individual selected at random has age=15?
2. Most probable age? 3. Median? $\langle j\rangle = \frac{\sum j N(j)}{N} = \sum_{j=0}^{\infty} j P(j)$
4. Average = expectation value of repeated measurements of many identically prepared system:
5. Average of squares of ages = $\langle j^2\rangle = \frac{\sum j^2 N(j)}{N} = \sum_{j=0}^{\infty} j^2 P(j)$
6. Standard deviation σ will give us uncertainty principle...

① There are 14 people and one is 15 years old so $P_{15} = 1/14$.

② There are 4 people age 25 - this is the most probable age

③ There is the same number of people above and below the median age - 7 above 23 and 7 below 23, so 23 is the median.

④ Average $= (1\times14 + 1\times15 + 3\times16 + 2\times22 + 2\times24 + 5\times25)/14 = 21 = \langle j\rangle$

⑤ $\langle j^2\rangle = (1\cdot14^2 + 1\cdot15^2 + 3\cdot16^2 + 2\cdot22^2 + 2\cdot24^2 + 5\cdot25^2)/14 = 459.57$

$(196 + 225 + 768 + 968 + 1152 + 3125)/14$

$(6434)/14$

Exercises in probability: uncertainty

$\langle j \rangle = 21$

$n = 1 \quad 1 \quad 3 \qquad 2 \quad 2 \quad 5$

$j = 14 \quad 15 \quad 16 \qquad 22 \quad 24 \quad 25$

$\Delta j = -7 \; -6 \; -5 \qquad 1 \quad 3 \quad 4$

$\Delta j^2 = 49, 36, 25, \; 1, \; 9, 16$

Standard deviation σ can be found from the deviation from the average: $\Delta j = j - \langle j \rangle$

But the average deviation vanishes: $\langle \Delta j \rangle = 0$

So calculate the average of the square of the deviation: $\sigma^2 = \langle (\Delta j)^2 \rangle$

$\sigma^2 = \langle (j - \langle j \rangle)^2 \rangle = \langle j^2 - 2j\langle j \rangle - \langle j \rangle^2 \rangle \ldots$

Exercise: show that it is valid to calculate σ more easily by:

$$\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2$$

HW: Find these quantities for the exercise above.

$\sigma^2 = \langle (\Delta j)^2 \rangle =$ average of $(\Delta j)^2$

$14 \cdot \sigma^2 = 49 + 36 + 3 \cdot 25 + 2 + 2 \cdot 9 + 5 \cdot 16$

$= 49 + 36 + 75 + 2 + 18 + 80$

$\sigma^2 = (85 + 75 + 20 + 80)/14 = (160 + 100 = 260)/14$

$= \frac{130}{7} = 18{,}57$

$\langle j^2 \rangle = 459{,}57$

$\langle j \rangle^2 = 21^2 = 441$

$\langle j^2 \rangle - \langle j \rangle^2 = 18.57$ ✓

In general, $\sigma^2 = \langle (\Delta j)^2 \rangle = \sum (\Delta j)^2 P(j) = \sum (j - \langle j \rangle)^2 P(j)$

$= \sum (j^2 - 2j\langle j \rangle + \langle j \rangle^2) P(j)$

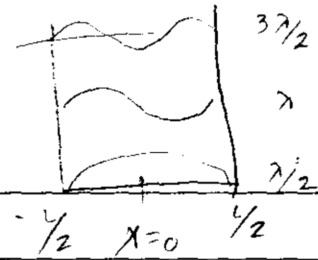
$= \sum j^2 P(j) - 2\langle j \rangle \sum j P(j) + \langle j \rangle^2 \sum P(j)$

$= \langle j^2 \rangle - 2\langle j \rangle \langle j \rangle + \langle j \rangle^2$

$\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2$

✓

Infinite square well centered on $x=0$:



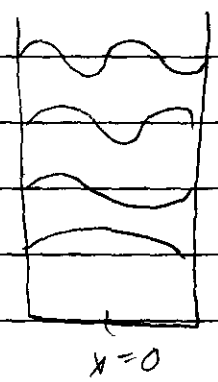
$L = \frac{n\lambda}{2} \rightarrow \lambda = \frac{2L}{n} \rightarrow \frac{1}{\lambda} = \frac{n}{2L}$

$pc = \frac{hc}{\lambda} \rightarrow p = \frac{h}{\lambda} = \frac{hn}{2L}$

If $V=0$ in the well, then $E = T = p^2/2m = \frac{n^2h^2}{2^2L^2 2m} = \frac{n^2h^2}{8mL^2}$

Same energy levels as for square well centered at $L/2$.

Wavefunctions can be both sines and cosines in this case, however,



$n=4,\ \lambda = \frac{L}{2},\ \sin kx,\ k = \frac{2\pi}{L/2} = 4\pi/L$

$n=3,\ \lambda = \frac{2}{3}L,\ \cos kx,\ k = \frac{2\pi}{2L/3} = \frac{3\pi}{L}$

$n=2,\ \lambda = L,\ \sin kx,\ k = \frac{2\pi}{\lambda} = \frac{2\pi}{L}$

$n=1,\ \lambda = 2L,\ \cos kx,\ k = \frac{2\pi}{\lambda} = \frac{2\pi}{2L} = \frac{\pi}{L}$

$K = \frac{n\pi}{L}$. For cos, $n =$ odd

For sin, $n =$ even

Show that stationary states are separable:

Guess that SE has separable solutions $\Psi(x,t) = \psi(x)\, f(t)$

$$\frac{\partial \Psi}{\partial t} = \psi(x)\frac{\partial f}{\partial t} \qquad \frac{\partial^2 \Psi}{\partial x^2} = f(t)\frac{\partial^2 \psi}{\partial x^2}$$

(a) sub into SE=Schrodinger Eqn $\quad i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$

(b) Divide by $\psi(x)\, f(t)$:

LHS(t) = RHS(x) = constant=E. Now solve each side:

(c) You already found solution to LHS: f(t)=_________

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

(d)

RHS solution depends on the form of the potential V(x).

(a) $$\frac{i\hbar\, \psi(x)\frac{\partial f(t)}{\partial t}}{\psi(x)\, f(t)} = \frac{-\hbar^2}{2m}\frac{f(t)\frac{\partial^2 \psi(x)}{\partial x^2}}{f(t)\,\psi(x)} + \frac{V\, f(t)\psi(x)}{f(t)\psi(x)}$$

(b)

$$\frac{i\hbar}{f(t)}\frac{\partial f(t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + V = E$$

(c) $$\frac{i\hbar}{f(t)}\frac{\partial f(t)}{dt} = E \rightarrow \frac{df}{f} = \frac{-iE}{\hbar}dt$$

$$\ln f = -iEt/\hbar + c$$

At $t=0$, $\ln f_0 = 0 + c \rightarrow \ln f = -iEt/\hbar + \ln f_0$

$$f = f_0 e^{-iEt/\hbar} \qquad \ln\left(\frac{f}{f_0}\right) = \frac{-iEt}{\hbar}$$

(d) So $\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$ where $\psi(x)$ is the solution

to $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$

Modern Physics Ch 6

7\.

A particle with mass m and total energy zero is in a particular region of space [...]re its wave function is $\psi(x) = Ce^{-x^2/L^2}$. (*a*) Find the potential energy $V(x)$ versus x and [m]ake a sketch of $V(x)$ versus x.

$$\frac{\partial\psi}{\partial x} = -\frac{2x}{L^2} Ce^{-x^2/L^2}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{-2C}{L^2}\left[x\left(-\frac{2x}{L^2}e^{-x^2/L^2}\right) + e^{-x^2/L^2}\left(\frac{\partial x}{\partial x}\right)\right] = \frac{-2C}{L^2}\left[1-\frac{2x^2}{L^2}\right]e^{-x^2/L^2}$$

$$= -\frac{2}{L^2}\left[1-\frac{2x^2}{L^2}\right]\psi$$

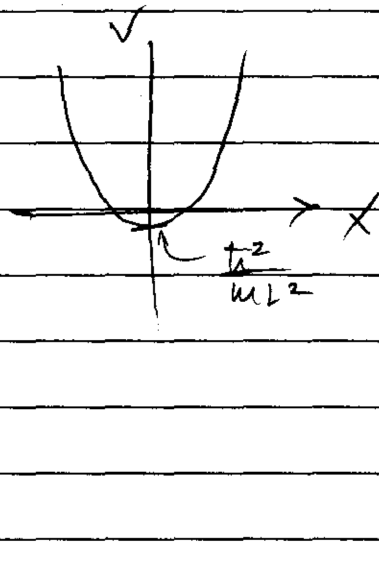
Schrödinger Eqn: $E\psi = \frac{p^2}{2m}\psi + V\psi$

$$0 = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$$ Sub in ψ'' and ψ:

$$\frac{\hbar^2}{2m}\left(\frac{-2}{L^2}\left[1-\frac{2x^2}{L^2}\right]\psi\right) = V\psi$$

Solve for ψ

$$\frac{\hbar^2}{mL^2}\left(\frac{2x^2}{L^2}-1\right) = V$$



19.

6-19. In the early days of nuclear physics before the neutron was discovered, it was thought that the nucleus contained only electrons and protons. If we consider the nucleus to be a one-dimensional infinite well with $L = 10$ fm and ignore relativity compute the ground-state energy for (*a*) an electron and (*b*) a proton in the nucleus. (*c*) Compute the energy difference between the ground state and the first excited state for each particle. (Differences between energy levels in nuclei are found to be typically of the order of 1 MeV.)

We found that the energy level of an infinite square well is $E_n = \frac{n^2 h^2}{8mL^2}$ (problem 5.7 : $E = KE$ if $V=0$)

For $L = 10 \times 10^{-15}$ m, $a = \frac{h^2}{8L^2} = \frac{(1240 \text{ MeV}\cdot\text{fm})^2}{8\,(10 \text{ fm})^2} = 1922 \text{ MeV}^2$

$E_n = \frac{n^2 a}{m}$

(a) For an electron, $E_{1_e} = \frac{a}{m_e = 0.511 \text{ MeV}} = \frac{1922 \text{ MeV}^2}{0.511 \text{ MeV}} = 3761 \text{ MeV}$

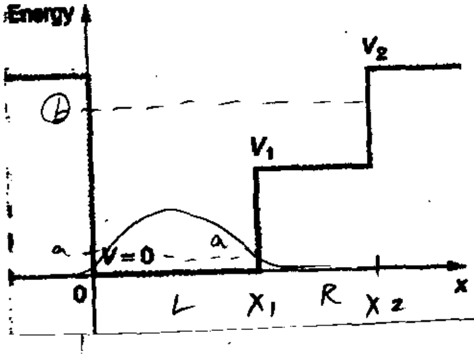
(b) For a proton, $E_{1_p} = \frac{a}{m_p = 938 \text{ MeV}} = \frac{1922 \text{ MeV}^2}{938 \text{ MeV}} = 2.05 \text{ MeV}$

(c) $(E_2 - E_1)_{electron} = (2^2 - 1^2) E_{1_e} = (4-1) E_{1_e} = 3 E_{1_e} = \underline{11.3 \text{ GeV}}$

$(E_2 - E_1)_{proton} = 3 E_{1_p} = \underline{6.15 \text{ MeV}}$

May seem surprising that electrons have higher energy! The energy of an object in an infinite well is $E \sim \frac{1}{mL^2}$

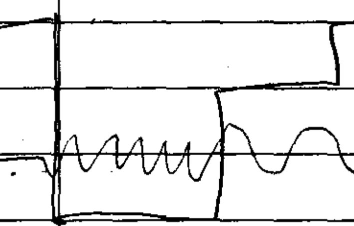
25.
Using arguments concerning curvature, wavelength, and amplitude, sketch very carefully the wave function corresponding to a particle with energy E in the finite potential well shown in Figure 6-33.



(a) If $E << V$, then wavefunctions nearly confined to the deep left well are allowed. These will have the highest amplitude, due to the normalization condition $1 = \int \psi^2 dx$

(b) For $V_1 < E < V_2$, notice that the kinetic energy is greater in the deep left well $T_L > T_R$ than in the shallow right well. ($E = T + V$ so $T = E - V$ is greater where V is low.)

Since $T = \frac{p^2}{2m}$ and $p = \frac{h}{\lambda}$, the wave length is larger where the kinetic energy is less



As we derived for the finite square well by considering the sign of the second derivative, the wave function MUST
- tunnel outside the well and
- vanish at infinity

27.

The mass of the deuteron (the nucleus of the hydrogen isotope ^2H) is 1.88 GeV/c^2. $= M$ How deep must a finite potential well be whose width is 2×10^{-15} m if there are two energy levels in the well? $= 2a$

See MORE link referenced on p. 263

$$\frac{\sin ka}{\cos ka} = \tan ka = \frac{\alpha}{k} \qquad \text{6-41}$$

Substituting values of k and α from above, Equation 6-41 can also be written as

$$\tan\left(\frac{\sqrt{2mE}}{\hbar}a\right) = \sqrt{\frac{V_0 - E}{E}} \qquad \text{6-42}$$

Considering the odd solutions in the well, $\psi(x) = A_1 \sin kx$, an equivalent discussion leads to the condition that

$$-\cot ka = \frac{\alpha}{k} \qquad \text{6-43}$$

Though tedious to solve analytically, the solutions to these transcendental equations can be readily found graphically. The solutions are those points where the graphs of

notice that this figure is missing from the text...

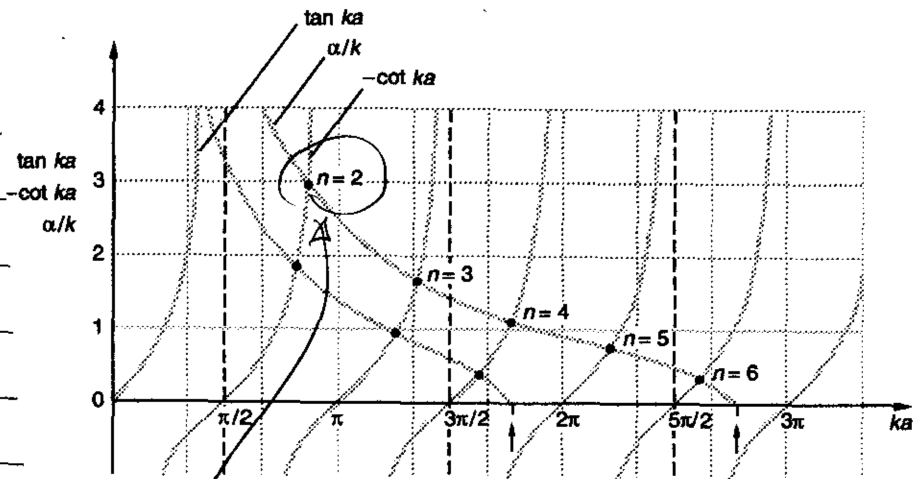


Fig. 6-14 Graphical solutions of Equations 6-41 and 6-43. Two different curves of α/k are shown, each corresponding to a different value of V_0. The value of V_0 in each case is given by the value of ka where $\alpha/k = 0$, indicated by the small arrows. For example, the top α/k curve has $\alpha/k = 0$ for $ka = 2.75\pi$, or $(2mV_0)^{1/2}a/\hbar = 2.75\pi$. Allowed values of E are those given by the values of ka at the intersections of the α/k and $\tan ka$ and α/k and $-\cot ka$ curves.

(Continued)

There will be at least two energy levels in the well if $ka > \frac{\pi}{2}$ where $ka = a\sqrt{2mE}/\hbar \rightarrow$ solve for $E = E_2$

$$\left(\frac{\pi}{2}\right)^2 = a^2 \frac{2mE}{\hbar^2} \rightarrow E = \frac{\pi^2 \hbar^2}{8a^2 m} = \dots = 25.6 \text{ MeV}$$