Vector Calculus (Physical Systems) **Final** – Tues.6.Mar.2007

This is a CLOSED-BOOK exam to be taken in class. You may use one page of your notes, plus the derivatives from the inside front cover of Griffiths (attached).
Please staple your notes to the exam when you hand it in.

You have 1 hour for this exam.
SHOW YOUR WORK to receive full credit, and include units whenever appropriate.
Please circle or underline your answers when appropriate, for clarity.
Express answers in *simplest exact form*.

(sign legibly) ZITA - SOLUTIONS

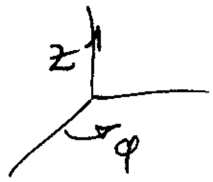
I affirm that I have worked this exam with WITHOUT using a calculator, text, HW, quizzes, computer, classmates, or other resources, except my one page of notes (attached).

1. (a) Can the divergence, gradient, or curl of the function below be found?

$$\mathbf{v} = s\left(2+sin^2\varphi\right)\hat{\mathbf{s}} + s\sin\varphi\cos\varphi\,\hat{\varphi} + 3z\,\hat{\mathbf{z}}$$

(b) **Which cannot be found? Why?**
gradient - can only be found for a scalar function

(c) In what coordinate system is the function **v** written? cylindrical



(c) **Find the divergence, gradient, and curl of the function, if possible.**

$$\bar\nabla\cdot\bar v = \frac{1}{s}\frac{\partial}{\partial s}\left(s^2(2+\sin^2\varphi)\right) + \frac{1}{s}\frac{\partial}{\partial\varphi}\, s\sin\varphi\cos\varphi + \frac{\partial}{\partial z}3z$$

$$= \frac{1}{s}2s(2+\sin^2\varphi) + \frac{s}{s}(\cos^2\varphi - \sin^2\varphi) + 3$$

$$= 2(2+\sin^2\varphi) + (\cos^2\varphi - \sin^2\varphi) + 3$$

$$= 4 + \cos^2\varphi + \sin^2\varphi + 3 = 4+1+3 = 8$$

$$(\bar\nabla\cdot\bar v)_s = \frac{1}{s}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z} = \frac{1}{s}\frac{\partial 3z}{\partial\varphi} - \frac{\partial}{\partial z}s\sin\varphi\cos\varphi = 0$$

$$(\bar\nabla\cdot\bar v)_\varphi = \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} = \frac{\partial}{\partial z}\left(s(2+\sin^2\varphi)\right) - \frac{\partial}{\partial s}(3z) = 0$$

$$(\bar\nabla\cdot\bar v)_z = \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\varphi) - \frac{\partial v_s}{\partial\varphi}\right]$$

$$\frac{\partial v_s}{\partial\varphi} = \frac{\partial}{\partial\varphi}s(2+\sin^2\varphi) = s2\sin\varphi\cos\varphi$$

$$\frac{\partial}{\partial s}(s v_\varphi) = \frac{\partial}{\partial s}s^2\sin\varphi\cos\varphi = 2s\sin\varphi\cos\varphi \rightarrow (\nabla\cdot v)_z = 0$$

$(\bar\nabla\times\bar v) = 0 \therefore \oint v\cdot dl = 0$: CONSERVATIVE FIELD (by the way)

2. (a) **Could the function $\mathbf{F} = y^2\,\hat{\mathbf{z}}$ be expressed as the gradient of a scalar?**

(b) **Explain why or why not.** Helmholtz theorem: If $\bar{\nabla}\times\bar{F} = 0 \Rightarrow \bar{F} = \bar{\nabla}u$

(c) **If so, find a suitable scalar** (show your method).

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & y^2 \end{vmatrix} = \hat{x}\left(\frac{\partial y^2}{\partial y} - \frac{\partial 0}{\partial z}\right) - \hat{y}\left(\frac{\partial y^2}{\partial x} - \frac{\partial 0}{\partial z}\right) + \hat{z}\left(\frac{\partial 0}{\partial x} - \frac{\partial 0}{\partial y}\right)$$

$$= \hat{x}\,(2y) \neq 0$$

No, $F \neq \nabla\bar{u}$

3. (a) **Could the function $\mathbf{F} = y^2\,\hat{\mathbf{z}}$ be expressed as the curl of a vector?**

(b) **Explain why or why not.** Helmholtz theorem: If $\bar{\nabla}\cdot\bar{F} = 0 \Rightarrow \vec{F} = \bar{\nabla}\times\bar{A}$

(c) **If so, find a suitable vector** (show your method).

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial y^2}{\partial z} = 0 + 0 + 0 = 0$$

yes, $$\vec{F} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$F_x = 0 \therefore \frac{\partial A_z}{\partial y} = \frac{\partial A_y}{\partial z}$

$F_y = 0 \therefore \frac{\partial A_z}{\partial x} = \frac{\partial A_x}{\partial z}$

$F_z = y^2 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$ $\rightarrow$ could have $A_x = 0$ and $A_y = \int y^2\,dx = y^2 x$, $A_z = 0$

or $A_y = 0$ and $A_x = -\int y^2\,dy = -\frac{y^3}{3}$, $A_z = 0$

4. Evaluate the following integrals:

(a) $\int_2^6 (3x^2-1)\delta(x-3)\,dx$ $= (3x^2-1)\Big|_{x=3} = 3\cdot 3^2 - 1 = 3\cdot 9 - 1 = 27-1 = 26$ ✓

(b) $\int_0^1 (\sin x)\delta\left(x-\frac{\pi}{2}\right)dx$ $= 0$ because $x = \frac{\pi}{2} \approx \frac{3.14}{2}$ is not in the interval $(0 \le x \le 1)$

(c) $\int_{-1}^1 (x+2)\delta(3x)\,dx$ $= \int_{-1}^{1} (x+2)\frac{1}{3}\delta(x)\,dx = \frac{1}{3}(x+2)\Big|_{x=0} = \frac{2}{3}$ ✓

(d) $\int_{-\infty}^{a} \delta(x-b)\,dx$ $= 1$ if $b<a$,
$x=+b$ $\quad = 0$ if $b>a$ (not in the interval $-\infty < x < a$)

5. (a) Write Stokes' theorem, or the fundamental theorem for curls.

$\oint \bar{v}\cdot d\bar{l} = \int (\bar{\nabla}\times\bar{v})\cdot d\bar{a}$
closed perimeter $\quad$ any open area bounded by the perimeter

**(b) What is the physical significance of a vector field with zero ~~divergence~~? curl
Give an example.**

If $\nabla\times\bar{v} = 0$ then $\oint \bar{v}\cdot d\bar{l} = 0$ and field is conservative
Ex: Electrostatic or gravitational field

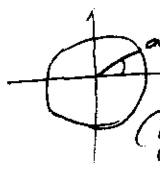
**(c) Check Stokes' theorem for the function $\mathbf{v} = 4y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}} + 2z\,\hat{\mathbf{z}}$ over the hemisphere $x^2+y^2+z^2=a^2$, above the plane $z\geq 0$.
Show your work explicitly for both sides of the equation. (Make it easy.)**

$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 4y & x & 2z \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}2z - \frac{\partial x}{\partial z}\right) - \hat{y}\left(\frac{\partial 2z}{\partial x} - \frac{\partial 4y}{\partial z}\right) + \hat{z}\left(\frac{\partial x}{\partial x} - \frac{\partial 4y}{\partial y}\right)$$
$$= \hat{x}(0) - \hat{y}(0) + \hat{z}(1-4)$$

$\bar{\nabla}\times\bar{v} = -3\hat{z}$ $\quad \int d\bar{a} =$ area of circle in x-y plane

$\int d\bar{a} = \pi a^2\,\hat{z}$

$\int(\bar{\nabla}\times\bar{v})\cdot d\bar{a} = -3\cdot\pi a^2$ ✓



$x = a\cos\phi \quad y = a\sin\phi$
$dx = -a\sin\phi\,d\phi,\ dy = a\cos\phi\,d\phi$
Cylindrical $d\bar{l} = a\,d\phi\,\hat{\phi}$

$\int \bar{v}\cdot d\bar{l}$ loops in a circle in the x-y plane

$= \int v_x\,dx + \int v_y\,dy = \int 4y\,dx + \int x\,dy$ $\quad (dz=0)$
$= \int 4a\sin\phi(-a\sin\phi\,d\phi) + \int a\cos\phi(a\cos\phi\,d\phi) = -4a^2\int_0^{2\pi}\sin^2\phi\,d\phi + a^2\int_0^{2\pi}\cos^2\phi\,d\phi$
$= -4a^2\pi + a^2\pi = -3a^2\pi$ ✓ $= \int(\bar{\nabla}\times\bar{v})\cdot d\bar{a}$ ✓ Stokes thm checks

Bjorn's spherical coordinates: loop around the xy plane has $d\bar{l} = r\sin\theta\, d\phi\, \hat{\phi}$ $(0 \le \phi < 2\pi)$

$\int \bar{v}\cdot d\bar{l} = \int v_\phi r\sin\theta\, d\phi = \int v_\phi\, a\, d\phi$ $\quad r=a,\ \theta=\frac{\pi}{2},\ \sin\frac{\pi}{2}=1$

$$\bar{v} = 4y\hat{x} + x\hat{y} + 2z\hat{z}$$
$$= 4y(\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi})$$
$$+ x(\sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi})$$
$$+ 2z(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$v_\phi = 4y(-\sin\phi) + x\cos\phi$$
$$= -4\sin\phi(r\sin\theta\sin\phi) + \cos\phi(r\sin\theta\cos\phi)$$
$$= -4r\sin\theta\sin^2\phi + r\sin\theta\cos^2\phi$$
$$= r\sin\theta(-4\sin^2\phi + \cos^2\phi)$$
$$-4\sin^2\phi + \cos^2\phi = -5\sin^2\phi + \sin^2\phi + \cos^2\phi = -5\sin^2\phi + 1$$

$v_\phi = r\sin\theta(1 - 5\sin^2\phi)$ at $r=a,\ \theta=\frac{\pi}{2}$ becomes

$$v_\phi = a(1 - 5\sin^2\phi)$$

$$\int \bar{v}\cdot d\bar{l} = a\int v_\phi\, d\phi = a^2\int (1 - 5\sin^2\phi)\, d\phi$$
$$= a^2\phi\Big|_0^{2\pi} - 5a^2\int \sin^2\phi\, d\phi$$
$$\int \sin^2\phi = \frac{\phi}{2} - \frac{\sin 2\phi}{4}\Big|_0^{2\pi}$$
$$= \frac{1}{2}(2\pi - 0) - \frac{1}{4}(\sin 4\pi - \sin 0) = \pi$$

$$\int \bar{v}\cdot d\bar{l} = a^2 2\pi - 5a^2\pi = -3a^2\pi$$
$$= \int (\nabla\times\bar{v})\cdot d\bar{a}$$