Vector Calculus HW #1 - Winter 2007 Physical Systems
E/Eta due Tues 23 Jan 07

Ch 1.1 Vector Algebra #5 p.8, 7 p.10, 10 p.12

#5 Prove the BAC-CAB rule by writing out both sides in component form: $\underbrace{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})}_{LHS} = \underbrace{\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})}_{RHS}$

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \hat{x}\begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix} - \hat{y}\begin{vmatrix} B_x & B_z \\ C_x & C_z \end{vmatrix} + \hat{z}\begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix}$$

$$= \hat{x}(B_yC_z - B_zC_y) - \hat{y}(B_xC_z - B_zC_x) + \hat{z}(B_xC_y - B_yC_x)$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}$$

$$= \hat{x}\begin{vmatrix} A_y & A_z \\ B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix} - \hat{y}\begin{vmatrix} A_x & A_z \\ B_yC_z - B_zC_y & B_xC_y - B_yC_x \end{vmatrix} + \hat{z}\begin{vmatrix} A_x & A_y \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z \end{vmatrix}$$

$$\begin{aligned} LHS &= \hat{x}\left[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)\right] \\ &- \hat{y}\left[A_x(B_xC_y - B_yC_x) - A_z(B_yC_z - B_zC_y)\right] \\ &+ \hat{z}\left[A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)\right] \end{aligned}$$

RHS = LHS ✓

1.7 Find $\bar{r} = \bar{r} - \bar{r}'$ from source $\bar{r}' = (2, 8, 7)$ to point $\bar{r} = (4, 6, 8)$

10 Find magnitude r and unit vector $\hat{r}$

$$\bar{r} = \bar{r} - \bar{r}' = (r_x - r_x')\hat{i} + (r_y - r_y')\hat{j} + (r_z - r_z')\hat{k}$$

$$= (4-2)\hat{i} + (6-8)\hat{j} + (8-7)\hat{k}$$

$$\underline{\bar{r} = 2\hat{i} - 2\hat{j} + \hat{k}}$$

$$|\bar{r}| = \underline{r} = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4+4+1} = \underline{3}$$

$$\underline{\hat{r}} = \frac{\bar{r}}{r} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

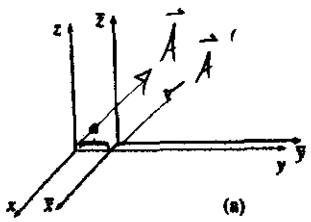
Problem 1.10

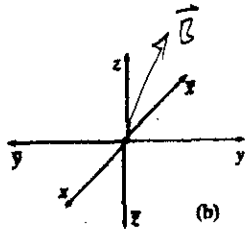
(a) How do the components of a vector transform under a **translation** of coordinates ($\bar{x} = x$, $\bar{y} = y - a$, $\bar{z} = z$, Fig. 1.16a)?

(b) How do the components of a vector transform under an **inversion** of coordinates ($\bar{x} = -x$, $\bar{y} = -y$, $\bar{z} = -z$, Fig. 1.16b)?

(c) How does the cross product (1.13) of two vectors transform under inversion? [The cross-product of two vectors is properly called a **pseudovector** because of this "anomalous" behavior.] Is the cross product of two pseudovectors a vector, or a pseudovector? Name two pseudovector quantities in classical mechanics.

(d) How does the scalar triple product of three vectors transform under inversions? (Such an object is called a **pseudoscalar**.)





ⓐ No change: vector's origin is not fixed; its direction and magnitude are.

vectors don't change under translation transformation

ⓑ $\vec{B}' = \ominus\vec{B}$: Same vector, negative coordinate labels

vectors change sign under inversion transformat.

ⓒ $(-\vec{A}) \times (-\vec{B}) = \vec{A} \times \vec{B}$: cross products (pseudovectors) DON'T change sign under inversion transformations

For example:
Angular momentum $\vec{L} = \vec{r} \times m\vec{v}$ and
torque $\vec{\tau} = \vec{r} \times \vec{F}$ are pseudovectors.

ⓓ $(-\vec{B}) \times (-\vec{C}) = \vec{B} \times \vec{C}$

$$-\vec{A} \cdot (-\vec{B} \times -\vec{C}) = -\vec{A} \cdot (\vec{B} \times \vec{C}) = -\left[\vec{A} \cdot (\vec{B} \times \vec{C})\right]$$

The triple product does change sign under inversion
Notice that this is a scalar, but a normal scalar does not change sign if coordinates invert, so we call this a PSEUDO SCALAR

Still EM HW#1 7. Oct. 02

27a

Ch 2 # $\frac{12}{75}$, $\frac{21}{82}$, $\frac{28}{87}$, $\frac{20}{79}$. We did half of these in class.

2.12: Find $\bar{E}$ inside a uniformly charged sphere of charge density
75 ρ = constant, total radius R.

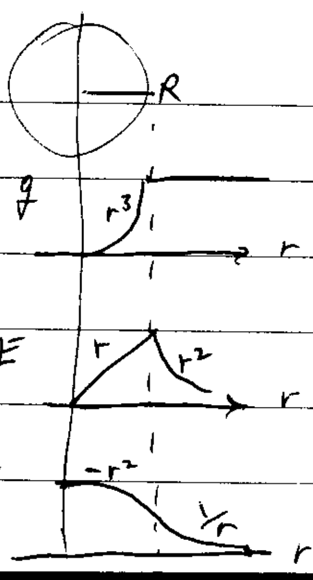
→ See attached solution: $q(r) = 4\pi\rho\frac{r^3}{3}$, $Q_{tot} = \frac{4}{3}\pi\rho R^3 = Q$

INSIDE: $\oint \bar{E}^{(r)} \cdot d\bar{a} = \frac{q(r)}{\epsilon_0}$: $\bar{E}(r<R) = \frac{\rho r}{3\epsilon_0}\hat{r}$

OUTSIDE: $\bar{E}(r \geq R) = \frac{\rho R}{3\epsilon_0}\hat{r} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$

2.21 Find the potential V inside & outside the same sphere of charge.
87 → See attached solution: Here are short answers

$\oint E \cdot dl = -\nabla V$: OUTSIDE: $V(r \geq R) = \frac{R^3\rho}{3\epsilon_0 r} = \frac{q}{4\pi\epsilon_0 r}$



INSIDE: $V(r<R) = \frac{\rho}{6\epsilon_0}(3R^2 - r^2)$

$= \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)$

<u>2.21</u> I can find $V = -\int \bar{E}\cdot d\bar{r}$ from $\bar{E}$ found in 2.12:

OUTSIDE: $V(r>R) = -\int_{\infty}^{r} E_{out}\, dr = -\frac{R^3\rho}{3\epsilon_0}\int_{\infty}^{r}\frac{dr}{r^2} = \frac{R^3\rho}{3\epsilon_0}\left(\frac{1}{r}-\frac{1}{\infty}\right)$

$$V_{out} = V(r>R) = \frac{R^3\rho}{3\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r}$$

At the surface: $V(R) = \frac{R^3\rho}{3\epsilon_0 R} = \frac{\rho R^2}{3\epsilon_0} = \frac{Q}{4\pi\epsilon_0 R}$

INSIDE:
$$V(r<R) = -\int_{\infty}^{R} E(r\geq R)\, dr - \int_{R}^{r} E_{in}\, dr$$
$$= V(R) - \int_{R}^{r}\frac{r\rho}{3\epsilon_0}\, dr$$
$$= \frac{R^2\rho}{3\epsilon_0} - \frac{\rho}{3\epsilon_0}\left(\frac{r^2}{2}-\frac{R^2}{2}\right)$$

$$V_{in} = V(r<R) = \frac{\rho}{6\epsilon_0}\left(2R^2+R^2-r^2\right) = \frac{\rho}{6\epsilon_0}\left(3R^2-r^2\right)$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3-\frac{r^2}{R^2}\right)$$

Check $\bar{E} = -\bar{\nabla} V$:

$$\nabla V_{in} = \hat{r}\frac{\partial}{\partial r}\left[\frac{\rho}{6\epsilon_0}\left(3R^2-r^2\right)\right] = -\frac{\rho r}{3\epsilon_0}\hat{r} = -\bar{E}_{in} \checkmark$$

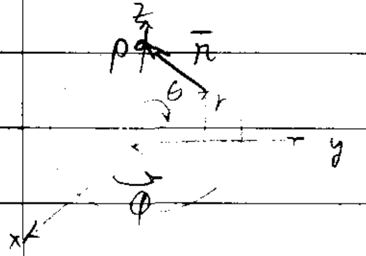
$$\nabla V_{out} = \hat{r}\frac{\partial}{\partial r}\left[\frac{R^3\rho}{3\epsilon_0 r}\right] = \frac{-R^3\rho}{3\epsilon_0 r^2}\hat{r} = -\bar{E}_{out} \checkmark$$

2.28 Calculate $V(r)$ again for the same sphere of charge.

87 This time use Eq. (2.29) (84): $V(\bar{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\bar{r}')}{\mathscr{r}}\, d\tau'$

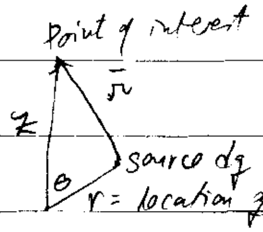
Consider a point on the z axis in spherical coordinates, for simplicity. By symmetry, $V(r)$ here is $V(r)$ everywhere.

$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}}\, d\tau$ where ρ = const.



volume element $d\tau = r^2 \sin\theta \, dr\, d\theta\, d\phi$ (p. 40)

$\bar{\mathscr{r}} = \bar{r} - \bar{r}'$ = point − source (p. 9)



See fig 2.33 (85), 1.36 (39)

r = location of source (drop r′ for notational convenience)

85 Law of cosines gives $\mathscr{r}^2 = r^2 + z^2 - 2rz\cos\theta$

Then $V = \frac{\rho}{4\pi\epsilon_0}\int \frac{d\tau}{\mathscr{r}} = \frac{\rho}{4\pi\epsilon_0}\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{r^2 \sin\theta\, dr\, d\theta\, d\phi}{\sqrt{r^2+z^2-2rz\cos\theta}}$

Notice that the integrand is independent of ϕ and $\int_0^{2\pi} d\phi = 2\pi$

So $V = \frac{\rho}{4\pi\epsilon_0} 2\pi \int_0^R\int_0^\pi \frac{r^2 \sin\theta\, dr\, d\theta}{\sqrt{r^2+z^2-2rz\cos\theta}}$. Do the θ integral for a given r:

Change variables: $I = \int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{r^2+z^2-2rz\cos\theta}} = \int_0^\pi \frac{d\alpha}{2rz}\frac{1}{\alpha^{1/2}} = \frac{1}{2rz}\left(2\alpha^{1/2}\right)\Big|_0^\pi$

Let $\alpha = r^2+z^2-2rz\cos\theta$

$d\alpha = -2rz(-\sin\theta)\, d\theta$

$\frac{d\alpha}{2rz} = \sin\theta\, d\theta$

$= \frac{1}{rz}\sqrt{r^2+z^2-2rz\cos\theta}\,\Big|_0^\pi$

$$I = \int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{z^2+r^2-2rz\cos\theta}} = \frac{1}{rz}\sqrt{r^2+z^2-2rz\cos\theta}\Big|_0^{\pi}$$ $\cos\pi = -1,\ \cos 0 = 1$

$$= \frac{1}{rz}\left[\sqrt{r^2+z^2+2rz} - \sqrt{r^2+z^2-2rz}\right]$$

$$I = \frac{1}{rz}\left(r+z-|r-z|\right) = \begin{cases} \frac{2}{z} & \text{if } r<z \\ \frac{2}{r} & \text{if } r>z \end{cases}$$

Return to $V = \frac{\rho}{4\pi\epsilon_0} 2\pi \int_0^R \int_0^{\pi} \frac{r^2 \sin\theta\, dr\, d\theta}{\sqrt{r^2+z^2-2rz\cos\theta}} = \frac{\rho}{2\epsilon_0}\int_0^R r^2 I\, dr$

$$= \frac{\rho}{2\epsilon_0}\int_0^z r^2\left(\frac{2}{z}\right)dr + \frac{\rho}{2\epsilon_0}\int_z^R r^2\left(\frac{2}{r}\right)dr$$

$(r<z)$ Sources inside point $\qquad$ $(r>z)$ Sources outside point

$$= \frac{\rho}{2\epsilon_0}\left[\frac{2}{z}\frac{r^3}{3}\Big|_0^z + 2\frac{r^2}{2}\Big|_z^R\right]$$

$$= \frac{\rho}{\epsilon_0}\left[\frac{1}{3z}(z^3-0) + \frac{1}{2}(R^2-z^2)\right]$$

$$= \frac{\rho}{\epsilon_0}\left[\frac{z^2}{3} + \frac{R^2}{2} - \frac{z^2}{2}\right]$$ $\frac{1}{3}-\frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}$

$$= \frac{\rho}{\epsilon_0}\left[\frac{R^2}{2} - \frac{z^2}{6}\right]$$

$V(r<R) = \frac{\rho}{6\epsilon_0}\left[3R^2 - z^2\right]$ SAME AS in 2.21

But this method was harder.

MORAL: Use Gauss' Law whenever possible. EASIER → fewer mistakes

2.20 Electrostatic fields satisfy $\bar{\nabla}\times\bar{E} = 0$ since $\frac{\partial B}{\partial t} = 0$.

79 Which of these fields cannot be an electrostatic field?

(a) $\bar{E} = k[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}]$

$$\bar{\nabla}\times\bar{E} = k\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = k\left(\begin{array}{l} \hat{i}\left(\frac{\partial}{\partial y}3xz - \frac{\partial}{\partial z}2yz\right) \\ -\hat{j}\left(\frac{\partial}{\partial x}3xz - \frac{\partial}{\partial z}xy\right) \\ +\hat{k}\left(\frac{\partial}{\partial x}2yz - \frac{\partial}{\partial y}xy\right) \end{array}\right)$$

$$\frac{\bar{\nabla}\times\bar{E}}{k} = \hat{i}(0-2y) - \hat{j}(3z-0) + \hat{k}(-x) \neq 0 \therefore \underline{\text{NOT}}\ \text{AN ELECTROSTATIC FIELD.}$$

(b) $\bar{E} = k[y^2\hat{x} + (2xy+z^2)\hat{y} + 2yz\hat{z}]$

$$\frac{\bar{\nabla}\times\bar{E}}{k} : \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & (2xy+z^2) & 2yz \end{vmatrix} = \begin{array}{l} \hat{i}\left(\frac{\partial}{\partial y}2yz - \frac{\partial}{\partial z}(2xy+z^2)\right) \\ -\hat{j}\left(\frac{\partial}{\partial x}2yz - \frac{\partial}{\partial z}y^2\right) \\ +\hat{k}\left(\frac{\partial}{\partial x}(2xy+z^2) - \frac{\partial}{\partial y}y^2\right) \end{array}$$

$$\bar{\nabla}\times\bar{E} = k\left[\hat{i}(2z-2z) - \hat{j}(0-0) + \hat{k}(2y-2y)\right] = 0\ \checkmark$$

This field is electrostatic.

(b)

2.20 continued - for the possible one, find the potential, using the origin as the reference point.

$$V(r) = -\int_0^r \vec{E}\cdot d\vec{l} = -\int E_x dx - \int E_y dy - \int E_z dz$$

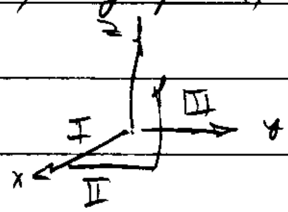
$$= k\left(-\int y^2 dx - \int (2xy + z^2) dy - \int 2yz\, dz\right)$$

$$= k\left(-y^2 x - \left(\frac{2xy^2}{2} + z^2 y\right) - \frac{2yz^2}{2}\right)$$

$$\frac{V}{k} = -y^2 x - xy^2 - z^2 y - yz^2 + C'$$

$$V = k(-2y^2 x - 2yz^2) - C$$

This is too high by a factor of 2. I'd better follow the instructions and PICK A PATH, any path, to integrate along. For example,



I: $y = z = 0$, $dy = dz = 0$, $x\Big|_0^{x_0}$

II: $x = x_0$, $y\Big|_0^{y_0}$, $z = dx = dz = 0$

III: $x = x_0$, $y = y_0$, $z\Big|_0^{z_0}$, $dx = dy = 0$

I $\quad V_I(r) = -\int E_x dx - \int E_y \cancel{dy}^{\,0} - E_z \cancel{dz}^{\,0} = -\int E_x dx = -k\int \cancel{y^2}^{\,0} dx = 0$

II $\quad V_{II}(r) = -\int E_x \cancel{dx}^{\,0} - \int k(2xy + z^2) dy - \int E_z \cancel{dz}^{\,0} = -k2x_0 \int_0^{y_0} y\, dy = -2kx_0 \frac{y_0^2}{2}$

$V_{II} = -kx$

III $\quad V_{III}(r) = -\int E_x \cancel{dx}^{\,0} - \int E_y \cancel{dy}^{\,0} - \int 2ky z\, dz = -2ky_0 \int_0^{z_0} z\, dz = -2ky_0 \frac{z_0^2}{2} = -k$

$V = -kxy^2 - kyz^2 + C$ / $-\frac{\partial V}{\partial x} = +ky^2$, $-\frac{\partial V}{\partial y} = 2kxy + kz^2$, $-\frac{\partial V}{\partial z} = 2kz$ ✓