Vector Calculus HW #2 due Tues 30 Jan 2007 - E/Z

1, 12, 13, 16, 25, 27

Problem 1.13 Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$. $\mathscr{r}^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$? $= n\mathscr{r}^{n-1}\hat{\mathscr{r}}$

(a) $\vec{\nabla}(\mathscr{r}^2) = \hat{x}\frac{\partial}{\partial x}\mathscr{r}^2 + \hat{y}\frac{\partial}{\partial y}\mathscr{r}^2 + \hat{z}\frac{\partial}{\partial z}\mathscr{r}^2$

$= \hat{x}\left[\frac{\partial}{\partial x}(x-x')^2\right] + \hat{y}\frac{\partial}{\partial y}(y-y')^2 + \hat{z}\frac{\partial}{\partial z}(z-z')^2$

$\frac{\partial}{\partial x}(x-x')^2 = \frac{\partial}{\partial x}(x^2 - 2xx' + x'^2) = 2x - 2x' = 2(x-x')$

as you can also see from the chain rule.

$\vec{\nabla}(\mathscr{r}^2) = \hat{x}\,2(x-x') + \hat{y}\,2(y-y') + \hat{z}\,2(z-z') = 2\vec{\mathscr{r}}$ ✓

vector

(b) $\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \vec{\nabla}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}$ [ ]

scalar

$= \frac{\partial}{\partial x}[\;]^{-1/2}\hat{x} + \frac{\partial}{\partial y}[\;]^{-1/2}\hat{y} + \frac{\partial}{\partial z}[\;]^{-1/2}\hat{z}$

$= -\frac{1}{2}[\;]^{-3/2}\frac{\partial}{\partial x}(x-x')^2\hat{x} - \frac{1}{2}[\;]^{-3/2}\frac{\partial}{\partial y}(y-y')^2\hat{y} - \frac{1}{2}[\;]^{-3/2}\frac{\partial}{\partial z}(z-z')^2\hat{z}$

$= -\frac{1}{2}[\;]^{-3/2}\{2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z}\}$

$= \dfrac{-\{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$

$= \dfrac{\vec{\mathscr{r}}}{[\mathscr{r}^2]^{3/2}} = \dfrac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \left(\dfrac{\vec{\mathscr{r}}}{\mathscr{r}}\right)\dfrac{1}{\mathscr{r}^2} = \dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

(c) $\frac{\partial}{\partial x}\mathscr{r}^n = n\mathscr{r}^{n-1}\frac{\partial \mathscr{r}}{\partial x} = n\mathscr{r}^{n-1}\frac{\mathscr{r}_x}{\mathscr{r}} = n\mathscr{r}^{n-1}\hat{\mathscr{r}}_x$

$\frac{\partial \mathscr{r}}{\partial x} = \frac{\partial}{\partial x}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2} = \frac{1}{2}\mathscr{r}^{-1/2}\,2(x-x') = \frac{\mathscr{r}_x}{\mathscr{r}}$

$\frac{\partial}{\partial y}\mathscr{r}^n = n\mathscr{r}^{n-1}\hat{\mathscr{r}}_y$, $\frac{\partial}{\partial z}\mathscr{r}^n = n\mathscr{r}^{n-1}\hat{\mathscr{r}}_z$

So $\vec{\nabla}\mathscr{r}^n = n\mathscr{r}^{n-1}\hat{\mathscr{r}}$

1.7 Find $\bar{r} = \bar{r} - \bar{r}'$ from source $\bar{r}' = (2,8,7)$ to point $\bar{r} = (4,6,8)$

10 Find magnitude r and unit vector $\hat{r}$

$$\bar{r} = \bar{r} - \bar{r}' = (r_x - r_x')\hat{i} + (r_y - r_y')\hat{j} + (r_z - r_z')\hat{k}$$
$$= (4-2)\hat{i} + (6-8)\hat{j} + (8-7)\hat{k}$$
$$\underline{\bar{r} = 2\hat{i} - 2\hat{j} + \hat{k}}$$

$$|\bar{r}| = \underline{r} = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4+4+1} = \underline{3}$$
$$\hat{r} = \frac{\bar{r}}{r} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

1.12 Height of hill (in feet) is $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$

15 See attached solution: Here are short answers.

(a) Top of hill where $dh = 0$: $x = -2$, $y = 3$: 3 mi N, 2 mi W of town

(b) Hill is 720' high

(c) Slope is $220\sqrt{2}\ \frac{ft}{mile}$ at $(x,y) = (1,1)$

(d) Direction of steepest slope is Northwest, in $-\hat{i} + \hat{j}$ direction.

(a) Solution: Slope = 0 at top of hill : $dh = 0 = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy \quad (\bar{\nabla}h = 0)$

$$\frac{\partial h}{\partial x} = 10(2y - 6x - 18 = 0) = 0 \text{ when } y - 3x = 9 \Rightarrow y = 9 + 3x$$
$$\frac{\partial h}{\partial y} = 10(2x - 8y + 28) = 0 \text{ when } x - 4y = -14$$

$$x = -14 + 4y = -14 + 4(9 + 3x) = -14 + 36 + 12x$$

$11x = 22 \Rightarrow \underline{x = -2}$, $\underline{y = 9 - 6 = 3}$ Top of hill at 3 mi N, 2 mi W of town

Problem 1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

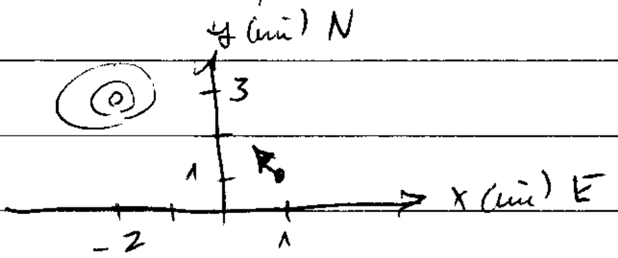
(b) How high is the hill? Evaluate h at top $= (-2, 3)$

$$h(x=-2, y=3) = 10(2xy - 3x^2 - 4y^2 - 18x - 28y + 12)$$

$$= 10(-4\cdot 3 - \cancel{3}\cdot 4 - \cancel{4}\cdot 9 + \cancel{18}\cdot 2 + 28\cdot 3 + \cancel{12})$$

height $= 10 \cdot 4(-3+21) = \underline{720 \text{ feet}}$

(c) How steep is slope $\left(\frac{\text{feet}}{\text{mile}}\right)$ at a point 1 mile N $(y=1)$ and 1 mile E $(x=1)$ from town



$$\nabla h = \frac{\partial h}{\partial x}\hat{\imath} + \frac{\partial h}{\partial y}\hat{\jmath} = 20\left[(y - 3x - 9)\hat{\imath} + (x - 4y + 14)\hat{\jmath}\right]$$

$$\nabla h(1,1) = 20\left[(1-3-9)\hat{\imath} + (1-4+14)\hat{\jmath}\right] = 20\cdot 11\left[-\hat{\imath} + \hat{\jmath}\right]$$

Slope: $|\nabla h| = 20 \cdot 11\sqrt{1^2 + 1^2} = 220\sqrt{2}$ ft/mi $\cong 311$ ft/mi

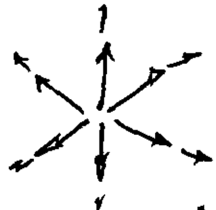
In what direction is slope steepest at $(1,1)$?
In $(-\hat{\imath} + \hat{\jmath})$ direction: Northwest

Problem 1.16 Sketch the vector function

p. 18

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}.$$

$r^2 = x^2 + y^2 + z^2$



and compute its divergence. The answer may surprise you... can you explain it?

Product Rule (iii) p. 21

$\vec{\nabla} \cdot f\vec{A} = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} f)$ Here $f = \frac{1}{r^2}$, $\vec{A} = \hat{r}$

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = \frac{1}{r^2}(\vec{\nabla} \cdot \hat{r}) + \hat{r} \cdot \left(\vec{\nabla} \frac{1}{r^2}\right)$$

$$\vec{\nabla} \cdot \hat{r} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (1 \cdot \hat{x} + 1 \cdot \hat{y} + 1 \cdot \hat{z}) =$$

$$= \frac{\partial}{\partial x} 1 + \frac{\partial}{\partial y} \cdot 1 + \frac{\partial}{\partial z} \cdot 1 = 0$$

$$\vec{\nabla} \frac{1}{r^2} = \hat{x}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1} + \hat{y}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1} + \hat{z}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1}$$

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1} = -(x^2+y^2+z^2)^{-2}(2x)$$

$$\vec{\nabla} \frac{1}{r^2} = \frac{-2x\hat{x} - 2y\hat{y} - 2z\hat{z}}{(x^2+y^2+z^2)^2} = \frac{-2(x\hat{x} + y\hat{y} + z\hat{z})}{r^4} = \frac{-2\vec{r}}{r^4} = -\frac{2\vec{r}}{r}\frac{1}{r^3}$$

$$\vec{\nabla} \frac{1}{r^2} = -\frac{2\hat{r}}{r^3} \quad \left(\text{spherical: } \vec{\nabla}\frac{1}{r^2} = \hat{r}\frac{\partial}{\partial r} r^{-2} = \hat{r}(-2 r^{-3}) = -\frac{2\hat{r}}{r^3}\right)$$

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = \frac{1}{r^2}(\vec{\nabla} \cdot \hat{r}) + \hat{r} \cdot \left(\vec{\nabla} \frac{1}{r^2}\right)$$

$$= 0 + \hat{r} \cdot \left(-\frac{2\hat{r}}{r^3}\right) = -\frac{2}{r^3} \to \infty \text{ at } r = 0$$

However, here are two derivations showing $\nabla \cdot \frac{\hat{r}}{r^2} = 0$:

1.16 What is $\nabla \cdot \left(\frac{\hat{r}}{r^2}\right)$? Look at Griffiths p. 50

Is $\nabla \frac{1}{r} = \frac{-\hat{r}}{r^2}$ as in M3.6?

$$\nabla \frac{1}{r} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}\,\hat{x} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}\,\hat{y} + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}\,\hat{z}$$

$$= -\tfrac{1}{2}(\ )^{-3/2}\,2x\,\hat{x} - \tfrac{1}{2}(\ )^{-3/2}\,2y\,\hat{y} - \tfrac{1}{2}(\ )^{-3/2}\,2z\,\hat{z}$$

$$= \frac{-(x\hat{x}+y\hat{y}+z\hat{z})}{(x^2+y^2+z^2)^{3/2}} = \frac{-\vec{r}}{r^3} = \frac{-\vec{r}}{r}\,\frac{1}{r^2} = \frac{-\hat{r}}{r^2} \checkmark$$

$$\nabla \frac{1}{r} = -\frac{\hat{r}}{r^2}$$

So $-\nabla \cdot \left(\nabla \frac{1}{r}\right) = \nabla \cdot \left(\frac{\hat{r}}{r^2}\right)$

$$\nabla \cdot (\nabla T) = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\nabla \cdot \left(\nabla \frac{1}{r}\right) = \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-1/2} + \frac{\partial^2}{\partial y^2}(x^2+y^2+z^2)^{-1/2} + \frac{\partial^2}{\partial z^2}(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-1/2} = \frac{\partial^2}{\partial x^2}(\ )^{-1/2}$$

$$= \frac{\partial}{\partial x}\left[-\tfrac{1}{2}(\ )^{-3/2}\,2x\right] = \frac{\partial}{\partial x}\left[-x(\ )^{-3/2}\right]$$

$$= -(\ )^{-3/2} + \tfrac{3}{2}x(\ )^{-5/2}\,2x = \frac{3x^2}{r^5} - \frac{1}{r^3}$$

$$\nabla \cdot \left(\nabla \frac{1}{r}\right) = \left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) + \left(\frac{3y^2}{r^5} - \frac{1}{r^3}\right) + \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)$$

$$= \frac{3(x^2+y^2+z^2)}{r^5} - \frac{3}{r^3} = \frac{3r^2}{r^5} - \frac{3}{r^3} = 0 = -\nabla \cdot \left(\frac{\hat{r}}{r^2}\right)$$

Check spherical: $\nabla \cdot (\nabla f) = \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} f\right)$

$$\nabla \cdot \left(\nabla \frac{1}{r}\right) = \nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \underbrace{\frac{\partial}{\partial r}\frac{1}{r}}_{-\frac{1}{r^2}}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-r^2}{r^2}\right) = 0 \checkmark$$

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{r\cdot r^2}\right)$$

$$\hat{x}\frac{\partial}{\partial x}\cdot\left(\frac{x\hat{x}}{r^3}\right) = \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial}{\partial x}x(x^2+y^2+z^2)^{-3/2}$$

$$= x\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial x}x$$

$$= x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) + (x^2+y^2+z^2)^{-3/2}$$

$$= \frac{-3x^2}{r^5} + \frac{1}{r^3}$$

Similarly, $\hat{y}\frac{\partial}{\partial y}\cdot\left(\frac{y\hat{y}}{r^3}\right) = -\frac{3y^2}{r^5} + \frac{1}{r^3}$

$$\hat{z}\frac{\partial}{\partial z}\cdot\left(\frac{z\hat{z}}{r^3}\right) = -\frac{3z^2}{r^5} + \frac{1}{r^3}$$

So $\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \left(\frac{-3x^2}{r^5}+\frac{1}{r^3}\right)+\left(-\frac{3y^2}{r^5}+\frac{1}{r^3}\right)+\left(-\frac{3z^2}{r^5}+\frac{1}{r^3}\right)$

$$= \frac{-3(x^2+y^2+z^2)}{r^5} + \frac{3}{r^3} = \frac{-3r^2}{r^5}+\frac{3}{r^3} = \frac{-3}{r^3}+\frac{3}{r^3}$$

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 0$$

Griffiths p. 50 says $\nabla\cdot\frac{\hat{r}}{r^2} = 0$ except at origin

Arfken p. 81 : $\int \nabla\cdot\frac{\hat{r}}{r^2}\,d\tau = \begin{cases} 4\pi & \text{including origin} \\ 0 & \text{excluding origin}\end{cases}$

Problem 1.25 Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$.

(b) $T_b = \sin x \sin y \sin z$.

(c) $T_c = e^{-5x} \sin 4y \cos 3z$.

(d) $\mathbf{v} = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

p. 24

$$\nabla^2 f(x,y,z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

(a) $\nabla^2 T_a = \frac{\partial}{\partial x}(2x+2y) + \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial z}(3)$
$= 2 + 0 + 0 = 2$

(b) $\nabla^2 T_b = \frac{\partial}{\partial x}(\cos x \sin y \sin z) + \frac{\partial}{\partial y}(\sin x \cos y \sin z) + \frac{\partial}{\partial z}(\sin x \sin y \cos z)$
$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$
$= -3T_b$

(c) $\nabla^2 T_c = \frac{\partial}{\partial x}(-5e^{-5x}\sin 4y \cos 3z) + 4\frac{\partial}{\partial y}(e^{-5x}\cos 4y \cos 3z) - 3\frac{\partial}{\partial z}(e^{-5x}\sin 4y \sin 3z)$
$= +25e^{-5x}\sin 4y \cos 3z - 16e^{-5x}\sin 4y \cos 3z - 9e^{-5x}\sin 4y \cos 3z$
$= (25-16-9)T_c = 0$

(d) $\nabla^2 \mathbf{v} = (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$

$$\nabla^2 v_x = \nabla^2(x^2) = \frac{\partial^2}{\partial x^2}(x^2) = \frac{\partial}{\partial x} 2x = 2$$

$$\nabla^2 v_y = \nabla^2(3xz^2) = \frac{\partial}{\partial x}(3z^2) + \frac{\partial}{\partial z}(3\cdot 2xz) = 0 + 6x$$

$$\nabla^2 v_z = -2\left[\frac{\partial}{\partial x}(z) + \frac{\partial}{\partial z}(x)\right] = 0$$

$$\nabla^2 \mathbf{v} = 2\hat{x} + 6x\,\hat{y}$$

Problem 1.27 Prove that the curl of a gradient is always zero. *Check* it for function (b) in Prob. 1.11.

p. 24

$$\text{Prove } \nabla\times(\nabla f) = 0 \quad \text{(scalar } f\text{)}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$\nabla\times\nabla f = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$(\nabla\times\nabla f)_x = \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y} = 0$ by equality of cross-derivatives

$-(\nabla\times\nabla f)_y = \frac{\partial}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial x} = 0$ " " " "

$(\nabla\times\nabla f)_z = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = 0$ " " " "

1.11(b) p. 15 Check for $f(x,y,z) = e^x \sin(y)\ln(z)$

$$\frac{\partial f}{\partial x} = e^x \sin y \ln z, \quad \frac{\partial f}{\partial y} = e^x \cos y \ln z, \quad \frac{\partial f}{\partial z} = e^x \sin y \frac{1}{z}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = e^x \cos y \ln z = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = e^x \cos y \ln z \checkmark$$

$$\frac{\partial}{\partial z}\frac{\partial f}{\partial x} = \frac{1}{z}e^x \sin y = \frac{\partial}{\partial x}\frac{\partial f}{\partial z} = e^x \sin y \frac{1}{z} \checkmark$$

$$\frac{\partial}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{z}e^x \cos y = \frac{\partial}{\partial y}\frac{\partial f}{\partial z} = e^x \cos y \frac{1}{z} \checkmark$$

Cross-derivatives are equal, therefore $\nabla\times(\nabla f) = 0$ ✓