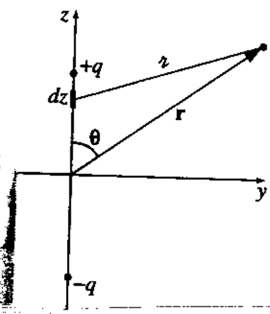
Thus 10 May 07
week 6

Ch 11 - Dipole Radiation - Do #1, 5

#1. **Problem 11.1** Check that the retarded potentials of an oscillating ELECTRIC dipole (Eqs. 11.12 and 11.17) satisfy the Lorentz gauge condition. Do *not* use approximation 3.

$\nabla\cdot A = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$ for Lorentz Gauge



Putting Eqs. 11.9 and 11.11 into Eq. 11.5, we obtain the potential of an oscillating perfect dipole:

$$V(r,\theta,t) = \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)] + \frac{1}{r}\cos[\omega(t-r/c)]\right\}. \qquad (11.12)$$

$$\boxed{\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)]\,\hat{\mathbf{z}}.}$$

Small sep
LF

[redacted]

V is in spherical coords, A is in cartesian

$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

$\frac{\partial V}{\partial t}$: $\frac{\partial}{\partial t}\sin\left[\omega t - \frac{\omega r}{c}\right] = \omega\cos\left(\omega t - \frac{\omega r}{c}\right)$

$\frac{\partial}{\partial t}\cos\left[\omega t - \frac{\omega r}{c}\right] = -\omega\sin\left(\omega t - \frac{\omega r}{c}\right)$

$\frac{\partial V}{\partial t} = \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega^2}{c}\cos - \frac{\omega}{r}\sin\right\}$

$k = \frac{-\mu_0 p_0 \omega}{4\pi}$

$A = \frac{k}{r}\sin\left(\omega t - \frac{\omega r}{c}\right)\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$ in spherical

$\nabla\cdot A_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{k}{r}\sin\left(\omega t - \frac{\omega r}{c}\right)\cos\theta\right] = \frac{k\cos\theta}{r^2}\frac{\partial}{\partial r}\left[r\sin\left(\omega t - \frac{\omega r}{c}\right)\right]$

$= \frac{k\cos\theta}{r^2}\left[\sin\sim - r\frac{\omega}{c}\cos\sim\right] = \frac{k}{r^2}\cos\theta\left[\sin\sim - \frac{\omega r}{c}\cos\sim\right]$

$\nabla\cdot A_\theta = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\,\frac{k}{r}\sin\left(\omega t - \frac{\omega r}{c}\right)(-\sin\theta)\right]$

$= \frac{-k\sin\left(\omega t - \frac{\omega r}{c}\right)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta = \frac{-k\sin\sim}{r^2\sin\theta}\,2\sin\theta\cos\theta$

$(\nabla \cdot A)_\theta = -\frac{2k}{r^2} \sin\sim \cos\theta$

$(\nabla \cdot A)_r = \frac{k}{r^2} \cos\theta \left[\sin\sim - \frac{\omega r}{c} \cos\sim \right]$

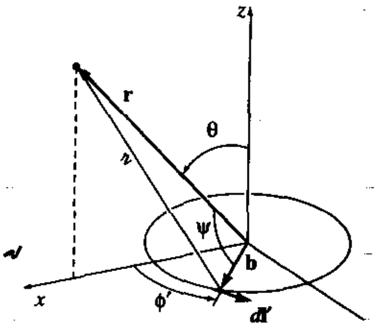
$$\nabla \cdot A = \frac{k}{r^2} \cos\theta \left[-\sin\left(\omega t - \frac{\omega r}{c}\right) - \frac{\omega r}{c} \cos\left(\omega t - \frac{\omega r}{c}\right) \right]$$

$$= \frac{\mu_0 p_0 \omega}{4\pi r^2} \cos\theta \left[\sin\left(\omega t - \frac{\omega r}{c}\right) + \frac{\omega r}{c} \cos\left(\omega t - \frac{\omega r}{c}\right) \right]$$

$$-\mu_0 \epsilon_0 \frac{\partial V}{\partial t} = -\frac{\mu_0 \epsilon_0 p_0 \cos\theta}{4\pi\epsilon_0 r} \left[-\frac{\omega^2}{c} \cos(\ ) - \frac{\omega}{r} \sin(\ ) \right]$$

$$= \nabla \cdot A \checkmark$$ $\therefore$ Lorentz Gauge.

Problem 11.5 Calculate the electric and magnetic fields of an oscillating magnetic dipol *without* using approximation 3. [Do they look familiar? Compare Prob. 9.33.] Find th Poynting vector, and show that the intensity of the radiation is exactly the same as we got usin approximation 3.



vector potential of an oscillating perfect magnetic dipole is

$$\mathbf{A}(r,\theta,t) = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{\frac{1}{r}\cos[\underbrace{\omega(t-r/c)}_{\sim}] - \frac{\omega}{c}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}. \qquad (11.33)$$

$$E = -\frac{\partial A}{\partial t} = \frac{\mu_0 m_0}{4\pi r}\left[\frac{1}{r}\left(+\frac{1}{c}\cos\sim\sin\sim\right) - \frac{\omega}{c}\left(-\frac{1}{c}\right)\sin\sim(\omega\sim\right.$$

$$= \frac{\mu_0 m_0}{4\pi r}\cos\sim\sin\sim\left(\frac{1}{rc} - \frac{\omega}{c^2}\right)\hat{\phi}$$

$$B = \nabla\times A = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta A_\varphi\right]\hat{r} - \left[\frac{1}{r}\frac{\partial}{\partial r} r A_\varphi\right]\hat{\theta}$$

$$rA_\varphi = \frac{\mu_0 m_0}{4\pi}\sin\theta\left\{\frac{1}{r}\cos\left(\omega t - \frac{\omega r}{c}\right) - \frac{\omega}{c}\sin\left(\omega t - \frac{\omega r}{c}\right)\right\}$$

$$\frac{\partial}{\partial r} rA_\varphi = \frac{\mu_0 m_0}{4\pi}\sin\theta\left\{-\frac{1}{r^2}\cos\sim + \frac{1}{r}\left(-\frac{\omega}{c}\right)(-\sin\sim) + \frac{\omega^2}{c^2}\cos\sim\right\}$$

$\hat{\theta}$ $$= \frac{\mu_0 m_0}{4\pi}\sin\theta\left\{\cos\sim\left(\frac{\omega^2}{c^2} - \frac{1}{r^2}\right) + \sin\sim\left(\frac{\omega}{rc}\right)\right\}$$

$$\sin\theta A_\varphi = \frac{\mu_0 m_0}{4\pi r}\left[\frac{1}{r}\cos\sim - \frac{\omega}{c}\sin\sim\right]\sin^2\theta$$

$$\frac{\partial}{\partial\theta}\sin\theta A_\varphi = \frac{\mu_0 m_0}{4\pi r}\left[\frac{1}{r}\cos\sim - \frac{\omega}{c}\sin\sim\right]2\sin\theta\cos\theta$$

$$B = \frac{2\mu_0 m_0}{4\pi r^2}\left[\frac{1}{r}\cos\sim - \frac{\omega}{c}\sin\sim\right]\hat{r} - \frac{\mu_0 m}{4\pi r}\sin\theta\left[\cos\sim\left(\frac{\omega^2}{c^2} - \frac{1}{r^2}\right) + \sin\sim\left(\frac{\omega}{rc}\right)\right]\hat{\theta}$$

$$S = \frac{1}{\mu_0}E\times B = \frac{1}{\mu_0}\begin{vmatrix}\hat{r} & \hat{\theta} & \hat{\phi}\\ E_r & E_\theta & E_\varphi\\ B_r & B_\theta & B_\varphi\end{vmatrix} = \frac{1}{\mu_0}\left[\hat{r}\left(-E_\varphi B_\theta\right) - \hat{\theta}\left(-E_\varphi B_r\right)\right]$$

Same as Prob 9.33 (p. 412) where we found that

$$S = \frac{\mu_0 m_0^2\omega^3}{16\pi^2 c^2}\left(\frac{\sin\theta}{r^2}\right)\left\{\frac{2\cos\theta}{r}\left[\left(1 - \frac{c^2}{\omega^2 r^2}\right)\sin u\cos u + \frac{c}{\omega r}\left(\cos^2 u - \sin^2 u\right)\right]\hat{\theta}\right.$$
$$\left.\sin\theta\left[\left(-\frac{2}{r} + \frac{c^2}{\omega^2 r^3}\right)\sin u\cos u + \frac{\omega}{c}\cos^2 u + \frac{c}{\omega r^2}\left(\sin^2 u - \cos^2 u\right)\right]\hat{r}\right\}$$

where $u = -\omega\left(t - \frac{r}{c}\right)$. Intensity is $\langle S\rangle = \frac{\mu_0 m_0^2\omega^4\sin^2\theta}{32\pi^2 c^3 r^2}\hat{r}$ same as (11.39)

Problem 11.6 Find the radiation resistance (Prob. 11.3) for the oscillating magnetic dipole
Fig. 11.8. Express your answer in terms of λ and b, and compare the radiation resistance
the *electric* dipole. [*Answer:* $3 \times 10^5\,(b/\lambda)^4\ \Omega$]

$$I^2R = I_0^2 R^2 \cos^2\omega t \rightarrow \langle P \rangle = \frac{1}{2} I_0^2 R = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} = \frac{\mu_0 \pi^2 b^4 I_0^2 \omega^4}{12\pi c^3}$$

$$R = \mu_0 \frac{\pi b^4 \omega^4}{6c^3}$$. Since $\omega = \frac{2\pi c}{\lambda}$,

$$R = \frac{\mu_0 \pi b^4}{6c^3}\frac{16\pi^4 c^4}{\lambda^4} = \frac{8}{3}\pi^5 \mu_0 c \left(\frac{b}{\lambda}\right)^4$$

$$= \frac{8}{3}\pi^5 \left(4\pi\times 10^{-7}\frac{Tm}{A}\right)\left(3\times 10^8 \frac{m}{s}\right)\left(\frac{b}{\lambda}\right)^4 \simeq 3\times 10^5 \left(\frac{b}{\lambda}\right)^4 \Omega$$

much smaller than for electric dipole.

Ex: $b = 5$ cm, $\lambda = 10^3$ m $\rightarrow R_B = 3\times 10^5 \left(5\times 10^{-5}\right)^4 = 2\times 10^{-12}\ \Omega$

$\sim 10^{-16} R_E$