EM 1b Physical Systems HW Thus.5.April 2007
Griffiths Ch.7 # 12, 17, 22, 25, 26, 28

Problem 7.12 A long solenoid, of radius a, is driven by an alternating current. so that the field inside is sinusoidal: $\mathbf{B}(t) = B_0 \cos(\omega t)\,\hat{\mathbf{z}}$. A circular loop of wire, of radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time. $\frac{d\Phi}{dt} = -\mathcal{E} = -IR = -\pi\left(\frac{a}{2}\right)^2(-\omega)B_0 \sin\omega t \rightarrow I(t) = \frac{\pi a^2 \omega B_0 \sin\omega t}{4R}$

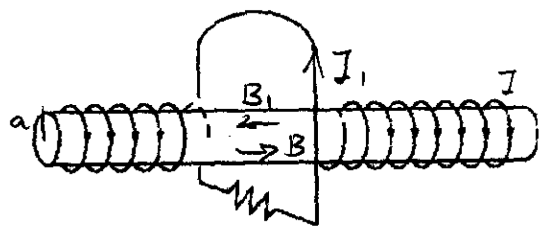
Problem 7.17 A long solenoid of radius a, carrying n turns per unit length, is looped by a wire with resistance R, as shown in Fig. 7.27. $\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\mu_0 n I A = -\mu_0 n k \pi a^2 = I_1 R$

$B = \mu_0 n I$

(a) If the current in the solenoid is increasing at a constant rate ($dI/dt = k$), what current flows in the loop, and which way (left or right) does it pass through the resistor? $I_1 = \frac{\mu_0 n k \pi a^2}{R}$

I_1 direction as shown, to drive B_1 opposite B

(b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop, turned around, and reinserted, what total charge passes through the resistor? Φ decreases then Φ increases, so the total charge $+Q - Q = 0$.



R

Problem 7.22 Find the self-inductance per unit length of a long solenoid, of radius R, carrying n turns per unit length. $B = \mu_0 n I$, $\Phi_1 = LI = \mu_0 n I \cdot A = \mu_0 n I \pi R^2$

$\Phi_n = \mu_0 n^2 I \pi R^2 \Rightarrow L = \mu_0 n^2 \pi R^2$

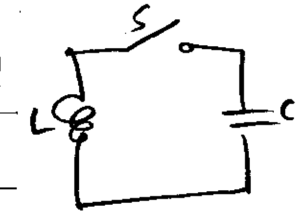
Problem 7.25 A capacitor C is charged up to a potential V and connected to an inductor L, as shown schematically in Fig. 7.38. At time $t = 0$ the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L? see p. 74 of my other notebook

Problem 7.26 Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length), (a) using Eq. 7.29 (you found L in Prob. 7.22); (b) using Eq. 7.30 (we worked out **A** in Ex. 5.12); (c) using Eq. 7.34; (d) using Eq. 7.33 (take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$).

see 7.41

Problem 7.28 A long cable carries current in one direction uniformly distributed over its (circular) cross section. The current returns along the surface (there is a very thin insulating sheath separating the currents). Find the self-inductance per unit length.

Problem 7.25 A capacitor C is charged up to a potential V_0 and connected to an inductor L, as shown schematically in Fig. 7.38. At time $t = 0$ the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L?

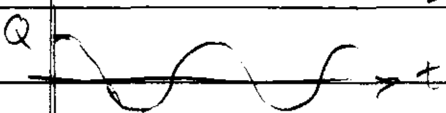


$$\Sigma V = 0 = V_C + V_L = \frac{Q}{C} + L\frac{dI}{dt}$$ (we did this in mechanics too)

$$\frac{1}{C}Q = -L\frac{d^2Q}{dt^2}$$ Guess $Q = Q_0 e^{i\omega t}$. Then $\ddot{Q} = -\omega^2 Q$

$$= +L\omega^2 Q$$

$$\frac{1}{LC} = \omega^2 \rightarrow \underline{Q = Q_0 \cos\omega t}$$ (since $Q(0) = Q_0 = CV$)



$$\underline{I = \frac{dQ}{dt} = -\omega Q_0 \sin\omega t}$$

If we add a resistor then $\Sigma V = V_L + V_R + V_C = L\frac{dI}{dt} + IR + \frac{Q}{C}$

$$0 = L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q$$

This is exactly like the damped harmonic oscillator (undriven)

which we solved in mechanics: $\Sigma F = m\ddot{x} = -kx - b\dot{x} = 0$ (CM lecture notes 24 Oct 02 EJZ)

$\omega_m^2 = \frac{k}{m}$, $2\gamma_m = \frac{b}{m}$ mechanical $\ddot{x} = -\omega_m^2 x - 2\gamma_m \dot{x}$

$\omega_L^2 = \frac{1}{LC}$, $2\gamma_L = \frac{R}{L}$ electrical $\ddot{Q} = -\frac{1}{LC}Q - \frac{R}{L}\dot{Q}$

$\sqrt{LC}$ = characteristic time for LC oscillations $\frac{L}{R}$ = decay time for RL circuit

The mechanical oscillator had solutions $x = e^{-\gamma t}(A_+ e^{qt} + A_- e^{-qt})$

where $q = i\omega_d$ and the damped frequency $\omega_d = \sqrt{\omega_m^2 - \gamma_m^2}$

So the electrical oscillator has solutions $Q = e^{-\gamma_L t}(A_+ e^{pt} + A_- e^{-pt})$

where $p = i\omega_{Ld}$ and $\omega_{Ld} = \sqrt{\omega_L^2 - \gamma_L^2}$

$$= \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

BOTTOM LINE: OSCILLATIONS DAMP

Apply IC: $V_c = V_0$, $I = 0$

$$Q = e^{-\gamma_L t}\left(A\cos\omega_{Lo}t + B\sin\omega_{Lo}t\right)$$

$$I = \frac{dQ}{dt} = e^{-\gamma_L t}\left[-\gamma_L\left(A\cos\omega_{Lo}t + B\sin_{Lo}t\right) + \left(-\omega_{Lo}A\sin + \omega_{Lo}B\,\mathrm{s}\right.\right.$$

$$I(0) = 0 \qquad \gamma_L\left(A\overset{1}{\cos} + \cancel{B\sin}\right) = \omega_{Lo}\left(\cancel{-A\sin} + B\overset{1}{\cos}\right)$$

$$\gamma_L A = \omega_{Lo} B$$

$$C = \frac{Q}{V} \rightarrow V = \frac{Q}{C}$$

$$V(t=0) = V_0 = \frac{1}{C}Q(t=0) = \frac{1}{C}e^0\left(A\cdot 1 + B\cdot 0\right)$$

$$V_0 = \frac{A}{C} \rightarrow \underline{A = CV_0}$$

$$B = \frac{\gamma_L}{\omega_{Lo}}A = \frac{\gamma_L C V_0}{\sqrt{\omega_L^2 - \gamma_L^2}} = \frac{\frac{2R}{L}CV_0}{\sqrt{\frac{1}{LC} - \left(\frac{2R}{L}\right)^2}} = \frac{2RCV_0}{\sqrt{\frac{L}{C} - 4R^2}} = \frac{2CV_0}{\sqrt{\frac{L}{R^2C} - 4}}$$

Check units: $\sqrt{\frac{L}{R^2C} - 4}$: $\frac{L}{R^2C} = \frac{L}{R}\cdot\frac{1}{RC} = \frac{\text{time}}{\text{time}}$

unitless ✓

B and A are both $\sim CV = Q$ ✓

Prob. 7.28
p. 320 (ed. 3)

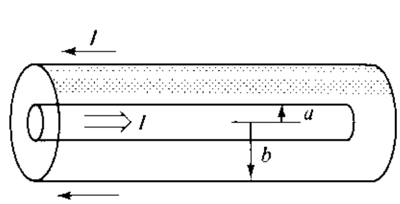


Figure 7.38

Solution: According to Ampère's law, the field between the cylinders is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\phi}$$

Elsewhere, the field is zero. Thus, the energy per unit volume is

$$\frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi r}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 r^2}$$

The energy in a cylindrical shell of length l, radius r, and thickness dr, then, is

$$\left(\frac{\mu_0 I^2}{8\pi^2 r^2}\right) l\, 2\pi r\, dr = \frac{\mu_0 I^2 l}{4\pi}\frac{dr}{r}$$

Integrating from a to b, we have:

$$W = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{b}{a}$$

By the way, this suggests a very simple way to calculate the self-inductance of the cable. According to (7.30), the energy can also be written as $\frac{1}{2}LI^2$. Comparing the two expressions,[8]

$$L = \frac{\mu_0 l}{2\pi} \ln \frac{b}{a}$$

(The self-inductance *per unit length* is $(\mu_0/2\pi)\ln(b/a)$.) This method of calculating the self-inductance is especially useful when the current is not confined to a single path but spreads over some surface or volume. In such cases different parts of the current may circle different amounts of flux, and it can be very difficult to get L from a formula like (7.25).

Problem 7.25 Find the energy stored in a section of length l of a long solenoid (radius R, current I, N turns per unit length),
(a) Using (7.30) (you found L in Problem 7.20).
(b) Using (7.31) (we worked out $\mathbf{A}$ in Example 12, Chapter 5).

[8]Notice the similarity to equation (7.27)—in a sense, the rectangular toroid is a short coaxial cable turned on its side.

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Problem 7.25 Find the energy stored in a section of length l of a long solenoid (radius R, current I, N turns per unit length),
(a) Using (7.30) (you found L in Problem 7.20). (c) All space
(b) Using (7.31) (we worked out **A** in Example 12, Chapter 5). (d) tube outside solenoid

In class I showed that solenoid field $\bar{B} = \mu_0 N I \hat{k}$

and self-inductance $\frac{L}{l} = \frac{\Phi N}{I} = \frac{\mu_0 N^2 I A}{I} = \mu_0 N^2 A = \mu_0 N^2 \pi R^2$

So the energy stored in the solenoid (per unit length)

(7.30) $\boxed{W = \frac{1}{2}LI^2}$ $\qquad \frac{W}{l} = \frac{1}{2}\frac{L}{l}I^2 = \frac{\mu_0}{2}\pi R^2 N^2 I^2$

(b) The solenoid vector potential from Ex 12 Ch 5 is

$\bar{A}_{in} = \frac{\mu_0 N I}{2} r\hat{\varphi}$, so using $\qquad W = \frac{1}{2}\oint_C (\mathbf{A}\cdot\mathbf{I})\, dl \qquad (7.31)$

$\underline{W} = \frac{1}{2}\int \bar{A}(R) I \underset{\substack{\uparrow \\ \#\text{ of turns}}}{n} \cdot d\bar{l} = \frac{1}{2}\frac{\mu_0 N I}{2} R I N l (2\pi R) = \underline{\frac{\mu_0}{2}\pi R^2 N^2 I^2 l}$

(c) if we agree to integrate over *all* space, then the surface integral goes to zero, and we are left with

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, d\tau} \qquad (7.35)$$

$B_{in} = \mu_0 N I$

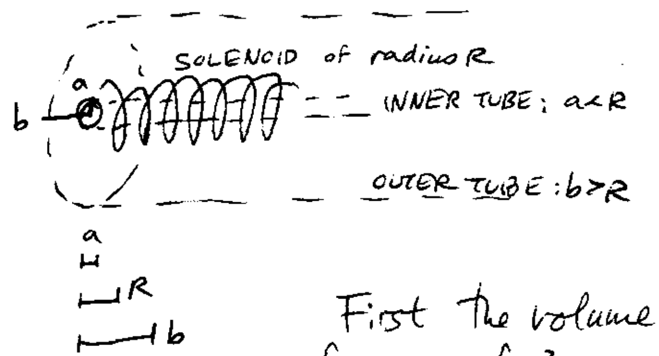
$B_{out} = 0$

$W = \frac{1}{2\mu_0} B_{in}^2 \cdot \text{volume} = \frac{1}{2\mu_0}(\mu_0 N I)^2 (\pi R^2 l) = \underline{\frac{\mu_0}{2}\pi R^2 N^2 I^2 l}$

(d)

$$W = \frac{1}{2\mu_0}\left[\int B^2\,d\tau - \int \nabla\cdot(\mathbf{A}\times\mathbf{B})\,d\tau\right]$$
$$= \frac{1}{2\mu_0}\left[\int_{\text{volume}} B^2\,d\tau - \oint_{\text{surface}} (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}\right] \tag{7.34}$$

Now, the integration (in 7.32) is to be taken over the *entire volume occupied by the current*. But any region *larger* than this will do just as well, for **J** is zero out there anyway. In the form (7.34), the larger the region we pick, the greater is the contribution from the B^2 integral and therefore the smaller is that of the surface integral (this makes sense: as the surface gets farther from the current, both **A** and **B** decrease). In

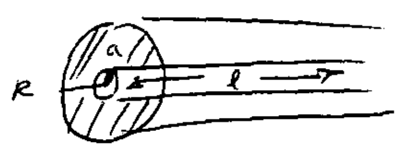


First the volume integral: $B_{out} = 0$

$$\int_{vol} B^2 d\tau = \int B_{in}^2\, d\tau_{in} + \int \cancel{B_{out}^2}\, d\tau_{out}$$

Volume of tube between $(a<r<R)$

$$\tau_{in} = \ell\,(\pi R^2 - \pi a^2)$$



$$\int_{vol} B^2 d\tau = (\mu_0 N I)^2 \ell\,(\pi R^2 - \pi a^2)$$

Now the surface integral, over surfaces at $r=a$ and $r=b$:

$$\oint_{surface} (\bar{A}\times\bar{B})\cdot da = \int^{r=b} (A_{out}\times \cancel{B_{out}}^{\,0})\cdot da_{out} + \int^{r=a} (\bar{A}_{in}\times\bar{B}_{in})\cdot da_{in}$$

$$= \int_{\phi=0}^{2\pi} \left(\frac{\mu_0 N I}{2}\, a\hat{\phi}\right)\times(\mu_0 N I \hat{k})\cdot(\ell a\, d\phi\, \hat{r}) = \frac{\mu_0^2 N^2 I^2}{2}\, 2\pi \ell a^2$$

these cancel

Finally, $W = \frac{1}{2\mu_0}\left[\int B^2 d\tau - \oint (A\times B)\cdot da\right] = \frac{1}{2\mu_0}\left[(\mu_0 N I)^2 \ell (\pi R^2 - \pi a^2) + (\mu_0 N I)^2 \ell \pi a^2\right]$

$$\frac{W}{\ell} = \frac{\mu_0}{2} N^2 I^2 \pi R^2$$