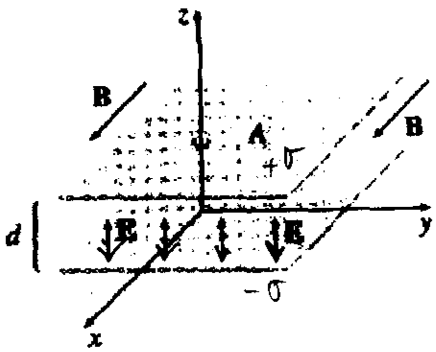
Physical Systems – spring 2007

EM HW 3a Tues.17 April 2007 Griffiths **Ch.8.2 # 5**

Problem 8.5 Consider an infinite parallel-plate capacitor, with the lower plate (at $z = -d/2$) carrying the charge density $-\sigma$, and the upper plate (at $z = +d/2$) carrying the charge density $+\sigma$.

(a) Determine all nine elements of the stress tensor, in the region between the plates. Display your answer as a 3×3 matrix:

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$



We know that $\mathbf{E} = -\frac{\sigma}{\epsilon_0}\hat{z}$ $(E_x = E_y = 0)$

If $\mathbf{J} = 0$ then $\mathbf{B} = 0$

Maxwell stress tensor,

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right). \tag{8.19}$$

are the same ($\delta_{xx} = \delta_{yy} = \delta_{zz} = 1$) and zero otherwise ($\delta_{xy} = \delta_{xz} = \delta_{yz} = 0$). Thus

$$T_{xx} = \frac{1}{2}\epsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2).$$

$$T_{xy} = \epsilon_0(E_x E_y) + \frac{1}{\mu_0}(B_x B_y),$$

$T_{xx} = \frac{1}{2}\epsilon_0\left(0 - 0 - \left(\frac{-\sigma}{\epsilon_0}\right)^2\right) + \frac{1}{2\mu_0}(0) = \frac{-\epsilon_0}{2}\frac{\sigma^2}{\epsilon_0^2} = \frac{-\sigma^2}{2\epsilon_0}$

$T_{yy} = \frac{1}{2}\epsilon_0\left(0 - 0 - \left(\frac{-\sigma}{\epsilon_0}\right)^2\right) + 0 = -\sigma^2/2\epsilon_0$

$T_{zz} = -\sigma^2/2\epsilon_0$

$T_{xy} = \epsilon_0(0 \cdot 0) + 0 = 0 = T_{yx}$

$T_{xz} = \epsilon_0(0 \cdot -\sigma/2\epsilon_0) + 0 = 0 = T_{zx} = T_{zy} = T_{yz}$

$$\overleftrightarrow{T} = \begin{pmatrix} \frac{-\sigma^2}{2\epsilon_0} & 0 & 0 \\ 0 & \frac{-\sigma^2}{2\epsilon_0} & 0 \\ 0 & 0 & \frac{-\sigma^2}{2\epsilon_0} \end{pmatrix}$$

(b) Use Eq. 8.22 to determine the force per unit area on the top plate. Compare Eq. 2.51.

$$(\nabla \cdot \overleftrightarrow{\mathbf{T}})_j = \epsilon_0 \left[(\nabla \cdot \mathbf{E})E_j + (\mathbf{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2 \right]$$
$$+ \frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})B_j + (\mathbf{B} \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2 \right]$$

Thus the force per unit volume (Eq. 8.18) can be written in the much simpler form

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0 \frac{\partial \mathbf{S}}{\partial t}, \tag{8.21}$$

The *total* force on the charges in $\mathcal{V}$ (Eq. 8.15) is evidently

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau. \tag{8.22}$$

$\mathbf{E} = -\frac{\sigma}{\epsilon_0}\hat{z}$ and $\mathbf{B} = 0$ so $\mathbf{S} = 0$

$\therefore$ $\mathbf{F} = \oint \overleftrightarrow{T} \cdot d\mathbf{a}$: For the top plate, use T_{zz} and integrate over the xy plane : $d\mathbf{a} = -dx\,dy\,\hat{z}$
outward

$F_z = \int T_{zz}\, da_z = \frac{-\sigma^2}{2\epsilon_0} A$

$\frac{\text{Force}}{\text{area}} = \mathbf{f} = \frac{-\sigma^2}{2\epsilon_0}\hat{z}$

(c) What is the momentum per unit area, per unit time, crossing the xy plane (or any other plane parallel to that one, between the plates)?

under (8.31) $\frac{\partial}{\partial t}\frac{\mathbf{p}}{\text{vol}} = \nabla \cdot \overleftrightarrow{T}$ we read that
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$-\overleftrightarrow{T}$ = momentum flux density

$-T_{ij}$ = momentum in i-direction crossing a surface oriented in the j-direction per unit area, per unit time

In the z-direction, $-T_{zz} = \frac{\sigma^2}{2\epsilon_0}$

(d) At the plates this momentum is absorbed, and the plates recoil (unless there is some nonelectrical force holding them in position). Find the recoil force per unit area on the top plate, and compare your answer to (b). [*Note:* This is not an *additional* force, but rather an alternative way of calculating the *same* force—in (b) we got it from the force law, and in (d) we did it by conservation of momentum.]

Recoil force $= \frac{dp}{dt}$ so $\frac{\text{force}}{\text{area}} = \frac{dp}{dt}\Big/\text{area} = -T = \frac{\sigma^2}{2\epsilon_0}\hat{z}$

Compare to

That argument applies to *any* surface charge; in the particular case of a conductor, the field is zero inside and $(\sigma/\epsilon_0)\hat{\mathbf{n}}$ outside (Eq. 2.48), so the average is $(\sigma/2\epsilon_0)\hat{\mathbf{n}}$, and the force per unit area is

$$\mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}}. \qquad (2.51)$$

✓ Same

This amounts to an outward electrostatic pressure on the surface, tending to draw the conductor into the field, regardless of the sign of σ. Expressing the pressure in terms of the field just outside the surface,

$$P = \frac{\epsilon_0}{2}E^2. \qquad (2.52)$$