Thus 10 May 07
week 6

QM 6b - Finish Ch 12.6 *12.6.1, 12.6.2
Ch 13 .2,3,4 H Atom *13.1.1, 13.1.3, 13.3.3

1.1 **Exercise 12.6.1.*** A particle is described by the wave function

$$\psi_E(r, \theta, \phi) = Ae^{-r/a_0} \qquad (a_0 = \text{const})$$

(i) What is the angular momentum content of the state?

(ii) Assuming ψ_E is an eigenstate in a potential that vanishes as $r \to \infty$, find E. (Match leading terms in Schrödinger's equation.) $E = -\hbar^2/2\mu a_0^2$

(iii) Having found E, consider finite r and find $V(r)$. $V = \hbar^2/\mu a_0 r$

(a) This wave function does not depend on θ or ϕ. The only Y_l^m that doesn't is $Y_0^0 = \sqrt{\frac{1}{4\pi}}$, so $l=0$ and $m=0$.
$L^2|\psi_E\rangle = l(l+1)|\psi_E\rangle = 0\,|\psi_E\rangle$; $L=0$

(b) This radial wavefunction must satisfy the radial Sch. eqn for some spherically symmetric potential $V(r)$:

(12.6.3) 349 $$\left\{\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] + V(r)\right\}R_{El} = E R_{El}$$

We know $l=0$: $$\left\{\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right] + V(r)\right\}\psi_E = E\psi_E$$

Do the $\frac{\partial}{\partial r}$ parts: $\frac{\partial}{\partial r}\psi_E = \frac{\partial}{\partial r}Ae^{-r/a_0} = -\frac{A}{a_0}e^{-r/a_0}$

$$\frac{\partial}{\partial r}r^2\frac{\partial\psi_E}{\partial r} = \frac{-A}{a_0}\frac{\partial}{\partial r}r^2e^{-r/a_0} = \frac{-A}{a_0}e^{-r/a_0}\left[\frac{-r^2}{a_0} + 2r\right]$$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}r^2\frac{\partial\psi_E}{\partial r}\right] = \frac{A}{a_0}e^{-r/a_0}\frac{1}{r^2}\left[2r - \frac{r^2}{a_0}\right] = \frac{A}{a_0}e^{-r/a_0}\left[\frac{2}{r} - \frac{1}{a_0}\right]$$

(12.6/...) So Sch. Eqn. becomes (sub in & simplify)

$$\frac{-\hbar^2}{2\mu}\frac{A}{a_0}e^{-r/a_0}\left[\frac{2}{r}-\frac{1}{a_0}\right]+V(r)Ae^{-r/a_0} = EAe^{-r/a_0}$$

$$\frac{-\hbar^2}{2\mu}\frac{1}{a_0}\left[\frac{2}{r}-\frac{1}{a_0}\right] = E-V$$

Let's look at the limits for infinite and finite r:

$$\lim(r\to\infty): \frac{-\hbar^2}{2\mu}\frac{1}{a_0}\left[\frac{-1}{a_0}\right] = E-V(r\to\infty)^{\nearrow 0} = E \to \frac{\hbar^2}{2\mu a_0^2}$$

Looks like an oscillator?

Sch. eqn. also has to work for finite r, and that will tell us V(r). Use the E we just found:

$$V = E + \frac{\hbar^2}{2\mu}\frac{1}{a_0}\left[\frac{2}{r}-\frac{1}{a_0}\right]$$

$$= \frac{-\hbar^2}{2\mu a_0^2}\left[\frac{-\hbar^2}{2\mu a_0}+\frac{\hbar^2}{\mu_0 a_0 r}\right]$$

$$V(r) = \frac{-\hbar^2}{\mu_0 a_0 r}$$

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Shankar

#12.6.2 - Derive (12.6.5) from (12.6.3)

(12.6.3) p.349 $\left\{ \frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] + V(r)\right\} R_{El} = E R_{El}$

Substitute for the radial wavefunction $R_{El} = \frac{U_{El}}{r}$, or $R = \frac{U}{r}$

$$\frac{\partial R}{\partial r} = \frac{\partial}{\partial r}\left(\frac{U}{r}\right) = \frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2}$$

$$r^2\frac{\partial R}{\partial r} = \frac{r^2}{r}\frac{\partial U}{\partial r} - \frac{r^2}{r^2}U = r\frac{\partial U}{\partial r} - U$$

$$\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} = \frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r} - U\right) = \frac{\partial U}{\partial r} + r\frac{\partial^2 U}{\partial r^2} - \frac{\partial U}{\partial r} = r\frac{\partial^2 U}{\partial r^2}$$

$$ER = E\frac{U}{r} = \left\{\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] + V(r)\right\}\frac{U}{r}$$

$$E\frac{U}{r} = \left\{-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2 U}{\partial r^2} + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\frac{U}{r} + V(r)\frac{U}{r}\right\}$$

Multiply both sides by $-2\mu r/\hbar^2$ to get

$$-\frac{2\mu r}{\hbar^2}E\frac{U}{r} = -\frac{2\mu}{\hbar^2}EU = \frac{\partial^2 U}{\partial r^2} - \frac{l(l+1)}{r^2}U - \frac{2\mu}{\hbar^2}V(r)U$$

$$\checkmark \quad 0 = \frac{\partial^2 U_{El}}{\partial r^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]U_{El}$$

$$\left\{\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]\right\}U_{El} = 0 \quad (12.6.5)$$