**QM HW 2b** Thus.12 April 2007 - **Shankar 5.1.2** (p.161, ed.1), **5.2.1** (p.163), **5.3.2** (p.176. Why? See 4.2.2)

*Exercise 5.1.2.** By solving the eigenvalue equation (5.1.3) in the $X$ basis, regain Eq. (5.1.8), i.e., show that the general solution of energy $E$ is

$$\psi_E(x) = \beta\,\frac{\exp[i(2mE)^{1/2}x/\hbar]}{(2\pi\hbar)^{1/2}} + \gamma\,\frac{\exp[-i(2mE)^{1/2}x/\hbar]}{(2\pi\hbar)^{1/2}}$$

[The factor $(2\pi\hbar)^{-1/2}$ is arbitrary and may be absorbed into $\beta$ and $\gamma$.] Though $\psi_E(x)$ will satisfy the equation even if $E<0$, are these functions in the Hilbert space?

$$H\,|E\rangle = \frac{P^2}{2m}\,|E\rangle = E\,|E\rangle \quad (5.1.3)$$ / eigenvalue eqn

$$|E\rangle = \beta\,|p=(2mE)^{1/2}\rangle + \gamma\,|p=-(2mE)^{1/2}\rangle \quad (5.1.8)$$

(5.1.3) 159 $\quad \underset{\text{energy operator}}{H}\,|E\rangle = \frac{P^2}{2m}\underset{\text{energy eigenstate}}{|E\rangle} = \underset{\text{energy eigenvalue}}{E}|E\rangle \longrightarrow \underset{\text{momentum operator}}{\frac{\left(i\hbar\frac{\partial}{\partial x}\right)^2}{2m}}\psi_E(x) = E\,\underset{\text{energy eigenstate in } x \text{ basis}}{\psi_E(x)}$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \longrightarrow \frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi = -\lambda^2\psi$$

This second order diffeq has solutions of the form

$\psi = A\cos\lambda x + B\sin\lambda x$ or

$\underline{\psi = Ae^{i\lambda x} + Be^{-i\lambda x}}$ ← we'll use this form.

$$\frac{\partial\psi}{\partial x} = +i\lambda Ae^{i\lambda x} - i\lambda Be^{-i\lambda x}$$

$$\frac{\partial^2\psi}{\partial x^2} = -\lambda^2 Ae^{i\lambda x} - \lambda^2 Be^{-i\lambda x} = -\lambda^2\psi \quad ✓$$

where $\lambda = \sqrt{2mE}/\hbar$

Our solution for $\psi$ is equivalent to Shankar's if we let $A = \frac{\beta}{\sqrt{2\pi\hbar}}$ and $B = \frac{\gamma}{\sqrt{2\pi\hbar}}$.

(He includes the $\frac{1}{\sqrt{2\pi\hbar}}$ to normalize $\psi$ to the $\delta$ function.)

*Exercise 5.2.1.** A particle is in the ground state of a box of length $L$. Suddenly the box expands (symmetrically) to twice its size, leaving the wave function undisturbed. Show that the probability of finding the particle in the ground state of the new box is $(8/3\pi)^2$.

Infinite square well of width $L$ has ground state $\psi_0 = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right)$ as we have shown before.
$(\psi = 0 \text{ for } |x| > \frac{L}{2})$

Similarly, for box of width $2L$, $\psi_0' = \sqrt{\frac{2}{2L}} \cos\left(\frac{\pi x}{2L}\right) = \sqrt{\frac{1}{L}} \cos\left(\frac{\pi x}{2L}\right)$
(and $\psi = 0$ for $|x| > L$)

Probability $(\psi \to \psi_0') = |\langle \psi_0 | \psi_0' \rangle|^2$

$$P^{1/2} = \langle \psi_0 | \psi' \rangle = \int_{-\infty}^{\infty} \psi_0^* \psi_0' \, dx = \int_{-L/2}^{L/2} \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) \sqrt{\frac{1}{L}} \cos\left(\frac{\pi x}{2L}\right) dx$$

since $\psi_0 = 0$ outside these limits

Let $\theta = \frac{\pi x}{L}$, $dx = \frac{L}{\pi} d\theta$. $\quad \theta(x = \frac{L}{2}) = \frac{\pi}{L}\frac{L}{2} = \frac{\pi}{2}$

Then $$P^{1/2} = \frac{\sqrt{2}}{L} \frac{L}{\pi} \int_{-\pi/2}^{\pi/2} \cos\theta \cos\frac{\theta}{2} \, d\theta = \frac{2\sqrt{2}}{\pi} \int_0^{\pi/2} \cos\theta \cos\frac{\theta}{2} \, d\theta$$

cos is even

Dwight 401, 06 $\quad 2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$$P^{1/2} = \frac{\sqrt{2}}{\pi} \int_0^{\pi/2} \cos\left(\theta + \frac{\theta}{2}\right) + \cos\left(\theta - \frac{\theta}{2}\right) d\theta = \frac{\sqrt{2}}{\pi} \left[ \int_0^{\pi/2} \cos\frac{3\theta}{2} \, d\theta + \int_0^{\pi/2} \cos\frac{\theta}{2} \, d\theta \right]$$

$$= \frac{\sqrt{2}}{\pi} \left[ \frac{2}{3} \sin\frac{3\theta}{2} \Big|_0^{\pi/2} + \frac{2}{1} \sin\frac{\theta}{2} \Big|_0^{\pi/2} \right] = \frac{\sqrt{2}}{\pi} \left[ \frac{2}{3} \sin\frac{3\pi}{4} + 2 \sin\frac{\pi}{4} \right]$$

$$= \frac{\sqrt{2}}{\pi} \left[ \frac{2}{3} \sqrt{\frac{1}{2}} + 2\sqrt{\frac{1}{2}} \right] = \frac{1}{\pi} \left[ \frac{2}{3} + \frac{6}{3} \right] = \frac{8}{3} \pi \to P = \frac{16}{9} \pi^2$$

*Exercise 5.2.1.** A particle is in the ground state of a box of length $L$. Suddenly the box expands (symmetrically) to twice its size, leaving the wave function undisturbed. Show that the probability of finding the particle in the ground state of the new box is $(8/3\pi)^2$.

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Probability $(\psi \to \psi_0') = |\langle \psi_0 | \psi_0' \rangle|^2$

$$P^{1/2} = \langle \psi_0 | \psi' \rangle = \int_{-\infty}^{\infty} \psi_0^* \psi_0' \, dx = \int_{-L/2}^{L/2} \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) \sqrt{\frac{1}{L}} \cos\left(\frac{\pi x}{2L}\right) dx$$

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Let $\theta = \frac{\pi x}{L}$, $dx = \frac{L}{\pi} d\theta$. $\theta(x = \frac{L}{2}) = \frac{\pi}{L} \frac{L}{2} = \frac{\pi}{2}$

$$\text{Then } P^{1/2} = \frac{\sqrt{2}}{L} \frac{L}{\pi} \int_{-\pi/2}^{\pi/2} \cos\theta \cos\frac{\theta}{2} \, d\theta = \frac{2\sqrt{2}}{\pi} \int_{0}^{\pi/2} \cos\theta \cos\frac{\theta}{2} \, d\theta$$

cos is even

Dwight 401.06 $\quad 2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$$P^{1/2} = \frac{\sqrt{2}}{\pi} \int_{0}^{\pi/2} \cos\left(\theta + \frac{\theta}{2}\right) + \cos\left(\theta - \frac{\theta}{2}\right) d\theta = \frac{\sqrt{2}}{\pi} \left[ \int_0^{\pi/2} \cos\frac{3\theta}{2} \, d\theta + \int_0^{\pi/2} \cos\frac{\theta}{2} \, d\theta \right]$$

$$= \frac{\sqrt{2}}{\pi} \left[ \frac{2}{3} \sin\frac{3\theta}{2} \Big|_0^{\pi/2} + \frac{2}{1} \sin\frac{\theta}{2} \Big|_0^{\pi/2} \right] = \frac{\sqrt{2}}{\pi} \left[ \frac{2}{3} \sin\frac{3\pi}{4} + 2 \sin\frac{\pi}{4} \right]$$

$$= \frac{\sqrt{2}}{\pi} \left[ \frac{2}{3} \sqrt{\frac{1}{2}} + 2\sqrt{\frac{1}{2}} \right] = \frac{1}{\pi} \left[ \frac{2}{3} + \frac{6}{3} \right] = \frac{8}{3} \pi \rightarrow P = \frac{16}{9} \pi^2$$

*Exercise 5.3.2.* Convince yourself that if $\psi = c\tilde{\psi}$, where $c$ is constant (real or complex) and $\tilde{\psi}$ is real, the corresponding $\mathbf{j}$ vanishes. cf 1.2.2

(5.3.8) 175 $$\mathbf{j} = \frac{\hbar}{2mi} \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*) = \text{probability current density}$$

If $\psi = c\tilde{\psi}$ then $\psi^* = c^* \tilde{\psi}$

$$\psi^* \nabla \psi = c^* \tilde{\psi} \nabla (c\tilde{\psi}) = c^* c\, \tilde{\psi} \nabla \tilde{\psi}$$

$$c^* c = (a - ib)(a + ib) = a^2 + b^2$$

$$\psi \nabla \psi^* = c\tilde{\psi} \nabla (c^* \tilde{\psi}) = cc^* \tilde{\psi} \nabla \tilde{\psi} = c^* c\, \tilde{\psi} \nabla \tilde{\psi}$$
$$= \psi^* \nabla \psi$$

Therefore $j = 0$

(5.3.9) $$\frac{d}{dt} \int P(r,t)\, d^3r = -\int_{S_\infty} j \cdot ds$$

If there is no probability flow into or out of the region, then there is no evolution in the probability $P = |\langle \psi | \psi \rangle|^2$

If $\int_{S_\infty} j \cdot ds = 0$ then the global probability for finding the particle anywhere in the universe is conserved.

If $j = 0$, we have the STRONGER constraint of LOCAL probability conservation.