Physical Systems – spring 2007

QM HW 2a Tues.10 April 2007 - Shankar 4.2.1 (p.135, ed.1), 4.2.2 (p.146), 4.2.3

Exercise 4.2.1 **(*Very Important*). Consider the following operators on a Hilbert space $V^3(C)$:**

$$L_x = \frac{1}{2^{1/2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}, \qquad L_y = \frac{1}{2^{1/2}}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix}, \qquad L_z = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$$

(a) What are the possible values one can obtain if L_z is measured?

Find eigenvalues from diagonal: $\omega_z = 1, 0, -1$

Let's find eigenvectors $|L_z\rangle = \begin{bmatrix}a\\b\\c\end{bmatrix}$ which satisfy $(L_z - \omega_z I)|L_z\rangle = 0$

$$\begin{bmatrix}1-\omega & 0 & 0\\ 0 & -\omega & 0\\ 0 & 0 & -1-\omega\end{bmatrix}$$

$\omega_z = 1$: $0 = \begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\-b\\-2c\end{bmatrix} \rightarrow a = \text{anything}$, $b = c = 0$ $\quad |L_z = 1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\omega_z = 0$: $0 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a\\0\\-c\end{bmatrix} \rightarrow b = \text{anything}$, $a = c = 0$ $\quad |L_z = 0\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$\omega_z = -1$: $0 = \begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}2a\\b\\0\end{bmatrix}$ $\quad b = a = 0 \rightarrow c = \text{anything}$ $\quad |L_z = -1\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$

(b) Take the state in which $L_z = 1$. In this state what are $\langle L_x\rangle$, $\langle L_x^2\rangle$, and ΔL_x?

$$\langle L_x\rangle = \langle L_z = 1|L_x|L_z = 1\rangle = [1\,0\,0]\frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}[1\,0\,0]\begin{bmatrix}0\\i\\0\end{bmatrix} = \underline{0 = \langle L_x\rangle}$$

←

(b)... In $|L_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, find $\langle L_x^2 \rangle$ and ΔL_x

$$L_x^2 = \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} +1 & 0 & -1 \\ 0 & +1 & 0 \\ -1 & 0 & +1 \end{bmatrix}$$

$$\langle L_x^2 \rangle = \langle L_z = 1 | L_x^2 | L_z = 1 \rangle = \frac{1}{2}[1\ 0\ 0]\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}[1\ 0\ 0]\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{2} = \underline{\langle L_x^2 \rangle}$$

$$\Delta L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} = \sqrt{\frac{1}{2} - 0} = \underline{\frac{1}{4} = \Delta L_x}$$

(c) Find the normalized eigenstates and the eigenvalues of L_x in L_z basis.

L_x: We're already in the L_z basis, so solve $\det(L_x - I\omega) = 0$

$$0 = \begin{vmatrix} -\omega & 1 & 0 \\ 1 & -\omega & 1 \\ 0 & 1 & -\omega \end{vmatrix} = -\omega(\omega^2 - 1) - 1(-\omega - 0) + 0$$

$$= -\omega(\omega^2 - 1 - 1) = -\omega(\omega^2 - 2)$$

$$= -\omega(\omega - \sqrt{2})(\omega + \sqrt{2}) \rightarrow \underline{\omega_x = 0, \pm\sqrt{2}}$$

eigenstates $|L_x\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfy $(L_x - \omega_x I)|L_x\rangle = 0$

$\omega_x = 0$: $0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ a+c \\ b \end{bmatrix} \rightarrow b = 0$, $a = -c$, $|L_x = 0\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} / \sqrt{2}$

$\omega_x = \sqrt{2}$: $0 = \begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \sqrt{2}a = b$, $b = \sqrt{2}c$, $|L_x = \sqrt{2}\rangle \rightarrow \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} / 2$

$\sqrt{1+2+1} = \sqrt{4} = 2$

$\omega_x = -\sqrt{2}$: $0 = \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \sqrt{2}a = -b$, $b = -\sqrt{2}c$, $|L_x = -\sqrt{2}\rangle = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} / 2$

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4.21 (d) If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities?

Possible L_x measurements are $\omega_x = 0, \pm\sqrt{2}$

$P(\omega_x = 0) = |\langle L_x = 0 | L_z = -1\rangle|^2 = \left|\frac{1}{\sqrt{2}}[1\ 0\ 1]\begin{bmatrix}0\\0\\1\end{bmatrix}\right|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$P(\omega_x = \sqrt{2}) = |\langle L_x = \sqrt{2} | L_z = -1\rangle|^2 = \left|\frac{1}{2}[1\ \sqrt{2}\ 1]\begin{bmatrix}0\\0\\1\end{bmatrix}\right|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$P(\omega_x = -\sqrt{2}) = |\langle L_x = -\sqrt{2} | L_x = -1\rangle|^2 = \left|\frac{1}{2}[1\ -\sqrt{2}\ 1]\begin{bmatrix}0\\0\\1\end{bmatrix}\right|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

(e) Consider the state

$$|\psi\rangle = \begin{bmatrix}1/2\\1/2\\1/2^{1/2}\end{bmatrix}$$

check norm $= \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right)^{1/2} = 1$ ✓

$L_z^2|\psi\rangle = 1|\psi'\rangle$

in the L_z basis. If L_z^2 is measured in this state and a result +1 is obtained, what is the state after the measurement? How probable was this result? If L_z is measured, what are the outcomes and respective probabilities?

$L_z^2 = \begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$

$L_z^2|\psi\rangle = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/2\\1/2\\1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}1/2\\0\\1/\sqrt{2}\end{bmatrix} = 1|\psi'\rangle$ where $|\psi'\rangle = \begin{bmatrix}1/2\\0\\1/\sqrt{2}\end{bmatrix}$

state after measurement

Probability of getting ψ' from ψ is $P = |\langle\psi'|\psi\rangle|^2$ normalized

norm of $\psi' = \left(\frac{1}{4} + \frac{1}{2}\right)^{1/2} = \sqrt{\frac{3}{4}} = \sqrt{3}/2$

$\psi'_{normalized} = \psi'_n = \begin{bmatrix}1/2\\0\\1/\sqrt{2}\end{bmatrix} \Big/ \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}}\begin{bmatrix}1/2\\0\\1/\sqrt{2}\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\\sqrt{2}\end{bmatrix}$ ✓

$P = |\langle\psi'_n|\psi\rangle|^2 = \frac{1}{3}\left|[1\ 0\ \sqrt{2}]\begin{bmatrix}1/2\\1/2\\1/\sqrt{2}\end{bmatrix}\right|^2 = \frac{1}{3}\left|\frac{1}{2} + 1\right|^2 = \frac{1}{3}\left|\frac{3}{2}\right|^2 = \frac{3}{4}$

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(e) continued... If $|\psi\rangle = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$ and L_z is measured,

the possible outcomes are the L_z eigenvalues, 0, 1, -1.
Their respective probabilities can be found from $P = |\langle\psi|L_z\rangle|^2$

$$P(L_z=0) = |\langle\psi|L_z=0\rangle|^2 = \left|\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$P(L_z=1) = \left|\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right|^2 = \frac{1}{4}$$

$$P(L_z=-1) = \left|\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right|^2 = \frac{1}{2}$$

(f) A particle is in a state for which the probabilities are $P(L_z = 1) = 1/4$, $P(L_z = 0) = 1/2$, and $P(L_z = -1) = 1/4$. Convince yourself that the most general, normalized state with this property is

$$|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z = 1\rangle + \frac{e^{i\delta_2}}{2^{1/2}}|L_z = 0\rangle + \frac{e^{i\delta_3}}{2}|L_z = -1\rangle$$

It was stated earlier on that if $|\psi\rangle$ is a normalized state then the state $e^{i\theta}|\psi\rangle$ is a physically equivalent normalized state. Does this mean that the factors $e^{i\delta_i}$ multiplying the L_z eigenstates are irrelevant? [Calculate for example $P(L_z = 0)$.]

Since $P(L_z=i) = |\langle\psi|L_z=i\rangle|^2 = P_i$ the simplest form would be (i = eigenvalues)

$$|\psi\rangle = \sum_i \sqrt{P_i}\,|L_z=i\rangle = \sqrt{\tfrac{1}{4}}|L_z=1\rangle + \sqrt{\tfrac{1}{2}}|L_z=0\rangle + \sqrt{\tfrac{1}{4}}|L_z=-1\rangle$$

$$|\psi\rangle = \frac{1}{2}|L_z=1\rangle + \frac{1}{\sqrt{2}}|L_z=0\rangle + \frac{1}{2}|L_z=-1\rangle$$

More generally, we are free to multiply by phase factors: $|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z=1\rangle + \frac{e^{i\delta_0}}{\sqrt{2}}|L_z=0\rangle + \frac{e^{i\delta_{-1}}}{2}|L_z=-1\rangle$

I showed in Moore SP4.1, however, that RELATIVE phases such as $(\delta_1-\delta_0)$ CAN affect the outcomes of measurements.

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