Physical Systems Final – EM & QM – Spring, Tues.29.May 2007 – TESC - EJZ

This is a take-home exam. You may use your notes, texts, and tables of integrals.
Each section is designed to take 2 hours; please limit yourself to 4 hours per section.

SHOW YOUR WORK neatly, and include units where appropriate, to receive full credit.
Use the back of the page if you need more space.
If you add extra pages, please staple them in, in order. Please, no scratch pages.
Please circle or underline your answers for clarity.
Express answers in *simplest exact form* whenever possible.
Order-of-magnitude estimates are usually fine for numerical problems.

(please sign legibly) ZITA - SOLUTIONS

I affirm that I have worked this exam using only my notes (including homework), texts, and table of integrals as resources – no calculators, computers, or other outside resources.

Electromagnetism: *Please note the time you spent on this section:* ____ *hours*

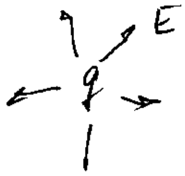
1. Maxwell equations and fundamental theorems	/ 22
2. Application (Faraday)	/ 10
3. Waves and Poynting vector	/ ~~12~~ 14
4. Relativity	/ 16
5. Reflection	/ 8
	Total = 70

Quantum Mechanics: *Please note the time you spent on this section:* ____ *hours*

1. Particle in a box	/ ~~20~~ 15
2. $\psi(t)$ and expectation values	/ ~~18~~ 22
3. Application (energy levels)	/ 13
4. Short answers	/ 20
	Total = 70

ELECTROMAGNETISM

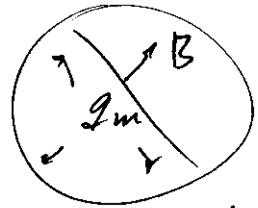
EM 1. **(a)** Write each **Maxwell equation** (there are four) in (22 pts)
(i) integral and (ii) differential form, with a (iii) clearly labeled diagram.
(16) (iv) For each equation, describe the causal relations (e.g. charge causes magnetic field?)



$$\oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$$

$$\nabla\cdot \underline{E} = \frac{\rho}{\epsilon_0}$$

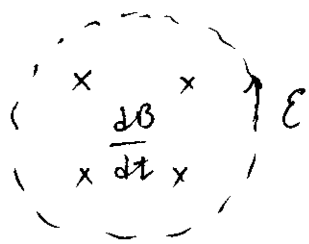
Gauss: charges cause $\underline{E}$ fields



$$\oint \vec{B}\cdot d\vec{A} = 0$$

$$\nabla\cdot \vec{B} = 0$$

no magnetic monopoles

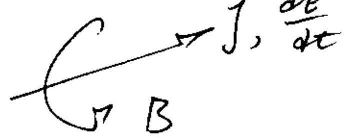


$$\oint E\cdot d\bar{l} = -\frac{d\Phi_B}{dt}$$

$$\nabla\times \underline{E} = -\frac{dB}{dt}$$

Faraday: changing magnetic fields induce electric fields

(as do potential gradients $\underline{E} = -\nabla V - \frac{\partial A}{\partial t}$)



Ampere: $\oint \bar{B}\cdot d\bar{l} = \mu_0 I + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$

$$\nabla\times B = \mu_0\left(J + \epsilon_0 \frac{dE}{dt}\right)$$

Currents & changing E fields cause magnetic fields

(3) **(b)** Derive the differential form of Gauss' law from the integral form, using the divergence theorem. Show your steps and define all terms.

$\oint E\cdot dA = \frac{q}{\epsilon_0}$ charge density $\rho = \frac{dq}{d\,\text{volume}} \rightarrow q = \int \rho\, d\tau$

Divergence theorem: $\oint_{\text{area}} E\cdot dA = \int_{\text{volume}} (\nabla\cdot E)\, d\tau = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho\, d\tau$

Equate integrands: (over the same volume) $\nabla\cdot E = \frac{\rho}{\epsilon_0}$ ✓

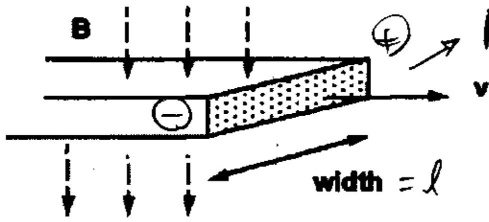
(3) **(c)** Derive the differential form of Ampere's law from the integral form, using the curl theorem. Show your steps and define all terms.

Ampere's law: $\oint B\cdot d\bar{l} = \mu_0 I + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$, current density $J = \frac{dI}{dA}$

Curl theorem: $\oint \bar{B}\cdot d\bar{l} = \int (\nabla\times B)\cdot dA = \mu_0\int J\cdot dA + \mu_0\epsilon_0 \frac{d}{dt}\int E\cdot dA$

Equate integrands: (through the same surface) $\nabla\times B = \mu_0 J + \mu_0\epsilon_0 \frac{dE}{dt}$

EM 2 – Application. A magnetohydrodynamic generator may consist of a rectangular channel within which flows a hot ionized gas, or plasma. Place the channel in a strong uniform magnetic field oriented as drawn below. (10 pts)



$\oplus \nearrow F = q\, v \times B$ magnetic force on charges separates charges, sets up Emf, attempts to drive plasma current across the channel

(2) (a) Consider the forces on the charged particles in the plasma. Electrons and ions can move relatively independently. **Draw** where the positive and negative particles will tend to move.

(3) (b) Derive an expression for the **electromotive force** induced between opposite sides of the stream of plasma.

(3) (c) **Find the magnitude of the emf** if the width of the channel is 50 cm, the speed of the plasma is 800 m/s, and the strength of the magnetic field is 6 Tesla. (That's a strong field!)

ⓑ $F_E = F_B$ induced Emf: $\left|E = -\frac{dV}{dx}\right| = \frac{\mathcal{E}}{\text{width}}$

$qE = qvB$

$E = vB$ $\mathcal{E} = vB \cdot \text{width}$

ⓒ $\mathcal{E} = Blv = 6\,T\,(\tfrac{1}{2}m)\,800\,\tfrac{m}{s} = 2400 \left(\frac{Tm^2}{s} = \text{volts}\right)$

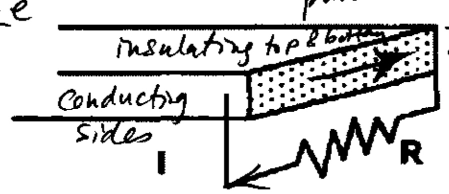
(2) (d) In which direction will current flow through a resistor connected as shown? **Draw and explain.**

$B\downarrow$ $\rightarrow I_p$ into page

$\downarrow B$

induced lorentz force $F = I\vec{l} \times \vec{B} \leftarrow \otimes I_p$

opposes the plasma current and the flux increase



MHD generators could serve as auxiliary generators in power plants burning fossil fuel. The MHD generator could use the exhaust gas released by the burning fuel as plasma. For this purpose, the fuel must be burned in a special high combustion chamber, similar to that of a jet engine, so as to produce a high-speed stream of very hot exhaust gas. After the hot gas emerges from the MHD generator, it could also supply heat to a boiler, providing steam for a conventional power generator.

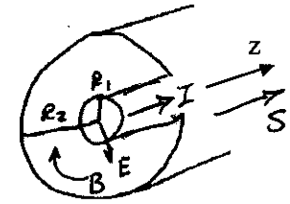
Optional challenge question: What questions would you ask to determine whether this scheme is cost effective? How much does it cost to maintain the strong B?

Are special materials needed to contain the plasma?

How much does it cost to run the high combustion chamber?

Is the plant lifetime reduced? Dangers?

EM 3 – Waves. Consider a simple coaxial waveguide. Assume that an electromagnetic wave propagates in the z direction, in the annular region between the two coaxial conductors, and there is zero field outside. Neglect any resistance of the conductors and assume a vacuum between them (though in practice there would be a dielectric). (12 pts)



As in a cylindrical capacitor, $\boldsymbol{E}$ is radial and inversely proportional to the radius. Since we have a wave, $E = \frac{K}{r} e^{i\left(\omega t - \frac{2\pi z}{\lambda}\right)}$, where K is a constant. Let $\boldsymbol{E}$ pointing outward be positive.

(1) (a) Recall the fundamental relationship between an electric field $\boldsymbol{E}$ and potential V. $E = -\frac{\partial V}{\partial r}\hat{r}$

(2) (b) Show that the **voltage of the inner conductor** with respect to the outer one is $V = K \ln \frac{R_2}{R_1} e^{i\left(\omega t - \frac{2\pi z}{\lambda}\right)}$. $V = -\int E \cdot dr$

(3) (c) Recall the simplest relationship between $\boldsymbol{E}$ and $\boldsymbol{B}$ in an electromagnetic plane wave. Use it to find $\boldsymbol{B}$ easily. Draw $\boldsymbol{B}$ on the diagram.

(2) (d) **Calculate** the current I on the inner conductor (hint: Ampere's law). **Draw** its direction, and **describe** it. (An equal current flows in the opposite direction along the outer conductor.)

(2) (e) Calculate the **Poynting vector**, and indicate its **direction.**

(2) (f) Integrate the Poynting vector over the annular region to calculate the time-averaged **transmitted power**.

(2) (g) Show that this is the same as the **Ohmic power**.

(b) $V = -\int E\,dr = -K e^{i(\omega t - 2\pi z/\lambda)} \int_{R_2}^{R_1} \frac{dr}{r} = -K e^{\sim} \ln r \Big|_{R_2}^{R_1} = K e^{\sim} \ln \frac{R_2}{R_1}$

(c) $B = \frac{\hat{k} \times E}{c} = \frac{\hat{z} \times E}{c} = \frac{E}{c}(\hat{z} \times \hat{r}) = \frac{E}{c}\hat{\theta} = \frac{k}{rc} e^{i\left(\omega t - \frac{2\pi z}{\lambda}\right)} \hat{\theta} = B$

(d) $\oint \bar{B} \cdot d\bar{l} = \mu_0 I = B \cdot 2\pi R_1 \rightarrow I = \frac{2\pi R_1 B}{\mu_0} = \frac{2\pi R_1}{\mu_0} \frac{k}{R_1 c} e^{\sim}$

$I = \frac{2\pi k}{\mu_0 c} e^{i\left(\omega t - \frac{2\pi z}{\lambda}\right)} \quad (-\hat{z})$

(e) $S = \frac{1}{\mu_0} E \times B = \frac{1}{\mu_0} \frac{E^2}{c} \hat{z} = \frac{\sqrt{\mu_0 \epsilon_0}}{\mu_0} \frac{K^2}{r^2} e^{2i\left(\omega t - \frac{2\pi z}{\lambda}\right)} \hat{z} = \sqrt{\frac{\epsilon_0}{\mu_0}} \frac{k^2}{r^2} e^{2i\left(\omega t - \frac{2\pi z}{\lambda}\right)}$

Lorraine & Corson 20-9 p.455

EM 3 continued (f) $S = \frac{\text{Power}}{\text{area}}$ so $\text{power} = \int S \cdot da$ where $da = 2\pi r\,dr$

$$\text{EM Power} = \int_{R_1}^{R_2} S_{AV} \cdot 2\pi r\,dr \quad \text{where} \quad S = \sqrt{\frac{\epsilon_0}{\mu_0}}\frac{k^2}{r^2} e^{2i(\omega t - \frac{2\pi z}{\lambda})}$$

$$S_{AV} = \text{time averaged } S = \frac{1}{2}|S| = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{k^2}{r^2}$$

$$\underline{\text{EM Power}} = \int_{R_1}^{R_2} \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{k^2}{r^2} 2\pi r\,dr = \sqrt{\frac{\epsilon_0}{\mu_0}} k^2 \pi \int_{R_1}^{R_2} \frac{dr}{r} = \underline{\sqrt{\frac{\epsilon_0}{\mu_0}} k^2 \pi \ln\frac{R_2}{R_1}}$$

watts

(g) $\text{Ohmic Power} = \langle VI \rangle = \frac{1}{2}|VI|$

$$VI = k \ln\frac{R_2}{R_1} e^{i(\omega t - \frac{2\pi z}{\lambda})} \cdot \frac{2\pi k}{\mu_0 c} e^{i(\omega t - \frac{2\pi z}{\lambda})}$$

$$= 2\pi k^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \ln\frac{R_2}{R_1} e^{2i(\omega t - \frac{2\pi z}{\lambda})}$$

$$\frac{1}{2}|VI| = \frac{2\pi k^2}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\ln\frac{R_2}{R_1} = \underline{\pi k^2 \sqrt{\frac{\epsilon_0}{\mu_0}} \ln\frac{R_2}{R_1} = \text{ohmic power}}$$

$= \text{EM power} \checkmark$

EM 4 – **Relativity** – short questions (16 pts)

(4) (a) If a particle's kinetic energy is *n* times its rest energy (mc^2), what is its speed? Griffiths 12.29 p.511

$$E = \gamma mc^2 = E_k + mc^2 \rightarrow E_k = mc^2(\gamma - 1) = n \cdot mc^2 \rightarrow \gamma = n+1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$1 - \frac{v^2}{c^2} = \frac{1}{(n+1)^2} \rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - 1}{(n+1)^2} = \frac{n^2+2n+1-1}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

$$\frac{v}{c} = \frac{\sqrt{n(n+2)}}{n+1}$$

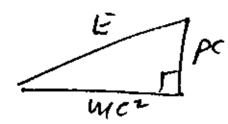
(6) (b) A neutral pion of rest mass *m* and (relativistic) momentum $p = \frac{3}{4}mc$ decays into two photons. One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. Find the (relativistic) energy of each pion. Griffiths 12.33 p.575

We could transform to the π rest frame, calculate E_γ (both the same), then transform back to lab frame, where $E_A \neq E_B$

$\pi \xrightarrow{v}$

B ← ∿∿ ∿∿∿ → A

Or we could apply energy & momentum conservation!



PION ENERGY IN LAB FRAME: $E_\pi^2 = (mc^2)^2 + (pc)^2 = (mc^2)^2[1^2 + (\frac{3}{4})^2] = (mc^2)^2[1 + \frac{9}{16} = \frac{25}{16}]$

$E_\pi = \frac{5}{4}mc^2 = E_A + E_B$ = sum of photon energies in lab frame

MOMENTUM CONSERVATION:

$$p_\pi = p_A + p_B$$
$$\frac{3}{4}mc = \frac{E_A}{c} - \frac{E_B}{c}$$
$$\frac{3}{4}mc^2 = E_A - E_B$$

$E = pc$

(photon B is going in the opposite direction)

combine with energy result:

$$\frac{5}{4}mc^2 = E_A + E_B$$

$$\frac{2}{4}mc^2 = 0 + 2E_B = \frac{1}{2}mc^2 \rightarrow \boxed{E_B = \frac{1}{4}mc^2}$$

$$E_A = \frac{3}{4}mc^2 + E_B = \boxed{mc^2 = E_A}$$

EM4 Relativity - continued

(6) (c) **Describe how electromagnetic fields transform relativistically, and how this accounts for the funny shape of the electric field of a very fast charge. Draw E and explain. Comment on B.**

$q \xrightarrow{v} x$ Charge at rest has a symmetric E field and $B = 0$.

Moving charge has unchanged field in the direction of motion : $\bar{E}_{\parallel} = E_{\parallel}$ $(\bar{E}_x = E_x)$

Perpendicular fields INCREASE : $\bar{E}_{\perp} = \gamma E_{\perp}$: $\bar{E}_y = \gamma E_y$, $\bar{E}_z = \gamma E_z$

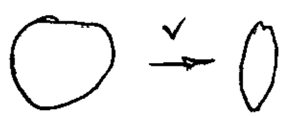
Thence the "pancake" shaped field of a fast-moving charge.

The magnetic field is nonzero in the moving frame.

Specifically (not required), the fields are given by (12.108) 531

$\bar{E}_x = E_x$, $\bar{E}_y = \gamma(E_y - vB_z) = \gamma E_y$, $\bar{E}_z = \gamma(E_z + vB_y) = \gamma E_z$

$\bar{B}_x = B_x = 0$, $\bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) = \frac{\gamma v}{c^2}E_z$, $\bar{B}_z = \gamma(B_z - \frac{vE_y}{c^2}) = -\frac{\gamma v}{c^2}E_y$

NB - distinguish this from the Lorentz Contraction of a fast moving sphere.  It takes a similar shape but the transformation is different : $\bar{E}_{\parallel} = \frac{E_{\parallel}}{\gamma}$

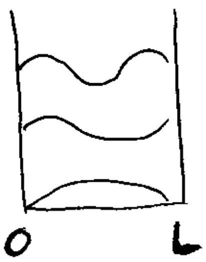
$\bar{E}_{\perp} = E_{\perp}$

QUANTUM MECHANICS

QM 1. **Particle in a 1D box**: (15 pts)

Consider a particle of mass m in an infinite square well of width L (from $0 \le x \le L$). The potential inside the box vanishes, and the potential outside the box is infinity.

(2) (a) What wavelengths fit inside the box? **Draw and label** appropriate **diagrams** for the potential, and use them to **derive** the quantization condition $\lambda_n =$ __________



$\frac{3}{2}\lambda_3 = L \rightarrow \lambda_3 = \frac{2}{3}L$

$\lambda_2 = L \rightarrow \lambda_2 = \frac{2}{2}L$

$\lambda_1 = 2L \rightarrow \lambda_1 = \frac{2}{1}L$

$\lambda_n = \frac{2L}{n}$

(1) (b) What is the deBroglie relationship between momentum and wavelength? $p(\lambda) = \hbar k = \frac{h}{\lambda}$

(2) (c) Use this to find the quantization condition on momentum $p_n = \frac{nh}{2L} = \frac{n\pi\hbar}{L}$

$p = \frac{h}{\lambda} = \frac{hn}{2L} = \frac{h}{2L} 2\pi\hbar = \frac{n\pi\hbar}{L}$

(1) (d) Write the Hamiltonian $H = \frac{p^2}{2m} + V = \frac{p^2}{2m}$ since $V = 0$

(1) (e) Find the relationship between energy and momentum for the particle: $E(p) = \frac{p^2}{2m}$

(2) (f) Use this to find the quantized energies $\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}}$

$E = \frac{p^2}{2m} = \frac{1}{2m}\left(\frac{n\pi\hbar}{L}\right)^2 = \frac{n^2\pi^2\hbar^2}{2mL^2}$ or $\frac{1}{2m}\left(\frac{nh}{2L}\right)^2 = \frac{n^2h^2}{8mL^2}$

(2) (g) Find the wavefunctions $\Psi_n(x) =$

$k = \frac{2\pi}{\lambda} = \frac{2\pi n}{2L} = \frac{n\pi}{L}$ $A\sin k_n x = A\sin\frac{n\pi x}{L} = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$

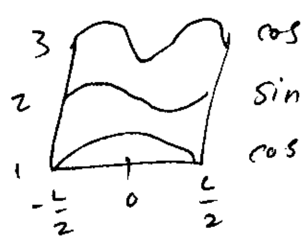
(2) (h) Find the time-dependent wavefunctions $\Psi_n(x,t) =$

$\Psi_n(x,t) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} e^{-iE_n t/\hbar} = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} e^{-in^2\pi^2\hbar t/2mL^2}$

(2) (g) How would your answers change if the well was centered at $x = 0$? EVERYTHING THE SAME

EXCEPT for odd n: $\Psi_n(x) \sim \cos\frac{n\pi x}{L}$

for even n: $\Psi_n(x) \sim \sin\frac{n\pi x}{L}$



QM 2. **Time evolution and expectation values:** (~~18~~ 22 pts) Chanian QM Ch 3 #5 p.93

Consider a particle of mass m in an infinite square-well potential of width L.

Its wavefunction at t=0 is $\psi(x,0)=\sqrt{\frac{2}{L}}\left[\frac{1}{\sqrt{3}}\sin\frac{2\pi x}{L}+\sqrt{\frac{2}{3}}\sin\frac{3\pi x}{L}\right] = A\left(a_2\psi_2(x)+a_3\psi_3(x)\right)$

(2) (a) Is this a stationary state? Explain. $\boxed{\text{No}}$ It's mixed, so it changes in space and time, no fixed nodes.

(2) (b) What is the energy associated with each eigenfunction in the mixed state?

As we showed in the previous problem, $\boxed{E=\frac{n^2\pi^2\hbar^2}{2mL^2}=n^2\varepsilon}$ $E_2=4\varepsilon$, $E_3=9\varepsilon$

(2) (c) What will be the wavefunction at a later time t?

(4) (d) What is the probability that a measurement of the energy at a time t will yield the value

(i) $\varepsilon=\frac{\hbar^2\pi^2}{2mL^2}$? 0 (ii) 4ε? $\frac{1}{3}$ (iii) 9ε? $\frac{2}{3}$ (iv) 16ε? 0

(5) (e) What is the expectation value of x at time t? $\langle x\rangle=\frac{L}{2}-\frac{\sqrt{2}\cdot 32}{25}\frac{L}{\pi^2}\cos\frac{5\varepsilon t}{\hbar}$ ✓

(5) (f) What is the expectation value of p at time t? $\langle p\rangle=\frac{16\sqrt{2}}{5}\frac{\hbar}{L}\sin\frac{5\varepsilon t}{\hbar}$ ✓

(2) (g) What is the frequency of oscillation of the wavefunction? $\boxed{\omega=\frac{5\varepsilon}{\hbar}}$

(c) $\psi(x,t)=\sqrt{\frac{2}{L}}\left[\frac{1}{\sqrt{3}}\sin\frac{2\pi x}{L}e^{-4\varepsilon it/\hbar}+\sqrt{\frac{2}{3}}\sin\frac{3\pi x}{L}e^{-9\varepsilon it/\hbar}\right]$

$n=2$ ($4=2^2$), $n=3$ ($9=3^2$)

(d) for $\psi(x,t)=A\sum_n a_n(t)\psi_n(x)$, the probability of measuring an energy value E_n is $P_n=|a_n|^2$

$n=2:\ E_2=4\varepsilon:\ P_2=|a_2|^2=\left|\frac{1}{\sqrt{3}}\right|^2e^{+4\varepsilon it/\hbar}e^{-4\varepsilon it/\hbar}=\frac{1}{3}e^0=\frac{1}{3}$ ✓

$n=3:\ E_3=9\varepsilon:\ P_3=|a_3|^2=\left|\sqrt{\frac{2}{3}}\right|^2e^{+9\varepsilon it/\hbar}e^{-9\varepsilon it/\hbar}=\frac{2}{3}e^0=\frac{2}{3}$ ✓

The other two are identically zero, since there are no ψ_1 or ψ_4 terms in $\psi(x,t)$.

(e) $\langle x\rangle=\int\psi^* x\,\psi\,dx$

$$=\frac{2}{L}\int_0^L\left(\sqrt{\tfrac{1}{3}}\psi_2(x)e^{4\varepsilon it/\hbar}+\sqrt{\tfrac{2}{3}}\psi_3(x)e^{9\varepsilon it/\hbar}\right)x\left(\sqrt{\tfrac{1}{3}}\psi_2(x)e^{-4\varepsilon it/\hbar}+\sqrt{\tfrac{2}{3}}\psi_3(x)e^{-9\varepsilon it/\hbar}\right)dx$$

$$=\frac{2}{L}\int_0^L\Big(\underbrace{\tfrac{1}{3}\psi_2^2(x)}_{I_1}+\underbrace{\tfrac{2}{3}\psi_3^2(x)}_{I_2}+\underbrace{\tfrac{\sqrt{2}}{3}\psi_2(x)\psi_3(x)\left[e^{5\varepsilon it/\hbar}+e^{-5\varepsilon it/\hbar}\right]}_{I_3}\Big)\,x\,dx$$

We have three integrals to evaluate:

QM 2.2ⓔ continued

$$\langle x\rangle = \frac{2}{L}\Big[\underbrace{\int_0^L \tfrac{1}{3}\, x\sin^2\frac{2\pi x}{L}\,dx}_{I_1} + \tfrac{2}{3}\underbrace{\int x\sin^2\frac{3\pi x}{L}\,dx}_{I_2} + \frac{\sqrt{2}}{3}\underbrace{\int x\sin\frac{2\pi x}{L}\sin\frac{3\pi x}{L}\,dx}_{I_3}\left(2\cos\frac{5\varepsilon t}{\hbar}\right)\Big]$$

$$I_3 = \int_0^L x\sin\frac{2\pi x}{L}\sin\frac{3\pi x}{L}\,dx$$

$$\sin A\sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin\frac{2\pi x}{L}\sin\frac{3\pi x}{L} = \tfrac{1}{2}\left[\cos\left(-\frac{\pi x}{L}\right) - \cos\left(\frac{5\pi x}{L}\right)\right]$$

$$= \tfrac{1}{2}\left[\cos\left(+\frac{\pi x}{L}\right) - \cos\left(\frac{5\pi x}{L}\right)\right]$$

$$I_3 = \tfrac{1}{2}\left[\int x\cos\frac{\pi x}{L}\,dx - \int x\cos\frac{5\pi x}{L}\,dx\right]$$

DWIGHT 440.11 p.101

$$\int y\cos y\,dy = \cos y + y\sin y$$

Let $y = ax$, $x = \frac{y}{a}$, $dx = \frac{dy}{a}$

$$\int x\cos ax\,dx = \int \frac{y}{a}\cos\frac{dy}{a} = \frac{1}{a^2}[\cos y + y\sin y]$$

$$\int_0^L x\cos\frac{\pi x}{L}\,dx = \left(\frac{L}{\pi}\right)^2[\cos y + y\sin y]_{y=0}^{\pi} = \left(\frac{L}{\pi}\right)^2[\cos\pi - \cos 0 + (\pi\sin\pi - 0)]$$

$$= \left(\frac{L}{\pi}\right)^2[-1-1+0] = \underline{-2\left(\frac{L}{\pi}\right)^2}$$

$$\int_0^L x\cos\frac{5\pi x}{L}\,dx = \left(\frac{L}{5\pi}\right)^2[\cos y + y\sin y]_0^{5\pi} = \left(\frac{L}{5\pi}\right)^2[\cos 5\pi - \cos 0 + (5\pi\sin 5\pi - 0)]$$

$$= \left(\frac{L}{5\pi}\right)^2[-1-1+0] = \underline{-2\left(\frac{L}{5\pi}\right)^2}$$

$$I_3 = -\frac{2}{2}\left[\left(\frac{L}{\pi}\right)^2 - \left(\frac{L}{5\pi}\right)^2\right] = -\left(\frac{L}{\pi}\right)^2\left[1 - \frac{1}{25} = \frac{24}{25}\right] = \underline{-\frac{24}{25}\left(\frac{L}{\pi}\right)^2}$$

For I_1 and I_2, $\int x\sin^2 ax\,dx$ Let $ax = y$, $x = \frac{y}{a}$, $dx = \frac{dy}{a}$

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$$\underline{\int x\sin^2 ax\,dx} = \left(\frac{1}{a}\right)^2\int y\sin^2 y\,dy = \underline{\frac{1}{4a^2}\left[y^2 - y\sin 2y - \tfrac{1}{2}\cos 2y\right]}$$

$$I_1 = \int_0^L x\sin^2\frac{2\pi x}{L}\,dx = \left(\frac{L}{2\pi}\right)^2\int_0^{2\pi} y\sin^2 y\,dy \quad \text{where } y = \frac{2\pi x}{L}$$

$$= \frac{1}{4}\left(\frac{L}{2\pi}\right)^2\left[y^2 - y\sin 2y - \tfrac{1}{2}\cos 2y\right]_0^{2\pi} = \frac{1}{4}\left(\frac{L}{2\pi}\right)^2\left[(2\pi)^2 - 0 - (2\pi\sin 4\pi - 0) - \tfrac{1}{2}(\cos 4\pi - \cos 0)\right]$$

$$= \frac{1}{4}\left(\frac{L}{2\pi}\right)^2\left[(2\pi)^2 - 2\pi\cdot 0 - \tfrac{1}{2}(0)\right] = \underline{\frac{L^2}{4}}$$

QM 2.2 (e) continued...

$$I_2 = \int_0^L x \sin^2 \frac{3\pi x}{L}\, dx = \left(\frac{L}{3\pi}\right)^2 \int_0^{3\pi} y \sin^2 y \, dy \quad \text{where } y = \frac{3\pi x}{L}$$

$$= \frac{1}{4}\left(\frac{L}{3\pi}\right)^2 \left[y^2 - y \sin 2y - \frac{1}{2}\cos 2y \right]_0^{3\pi}$$

$$= \frac{1}{4}\left(\frac{L}{3\pi}\right)^2 \left[(3\pi)^2 - 0 - (3\pi \sin 6\pi - 0) - \frac{1}{2}(\cos 6\pi - \cos 0)\right]$$

$$= \frac{1}{4}\left(\frac{L}{3\pi}\right)^2 \left[(3\pi)^2 - 0 \qquad - \frac{1}{2}(1-1)\right] = \underline{\frac{L^2}{4}}$$

$$\langle x \rangle = \frac{2}{L}\left[\frac{1}{3} I_1 + \frac{2}{3} I_2 + \frac{\sqrt{2}}{3} I_3 \, 2 \cos \frac{5\varepsilon t}{\hbar} \right]$$

$$\frac{1}{3} I_1 + \frac{2}{3} I_2 = \left(\frac{1}{3} + \frac{2}{3}\right) \frac{L^2}{4} = \frac{L^2}{4} = \left(\frac{L}{2}\right)^2$$

$$\langle x \rangle = \frac{2}{L}\left[\left(\frac{L}{2}\right)^2 + \frac{\sqrt{2}}{3}\left(-\frac{24}{25}\right)\left(\frac{L}{\pi}\right)^2 2 \cos \frac{5\varepsilon t}{\hbar} \right]$$

$$\boxed{\langle x \rangle = \frac{L}{2} - \frac{\sqrt{2} \cdot 32 L}{25 \pi^2} \cos \frac{5\varepsilon t}{\hbar}}$$

Wow, that was a lot of calculation. That deserves an extra 2 points at least.

(f) $\langle p \rangle = \int \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx$

$$= -i\hbar \frac{2}{L} \int \left(\frac{1}{\sqrt{3}} \sin \frac{2\pi x}{L} e^{4\varepsilon i t/\hbar} + \sqrt{\frac{2}{3}} \sin \frac{3\pi x}{L} e^{9\varepsilon i t/\hbar} \right) \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{3}} \sin \frac{2\pi x}{L} e^{-4\omega} + \sqrt{\frac{2}{3}} \sin \frac{3\pi x}{L} e^{-9\omega} \right) dx$$

$$= \frac{-2i\hbar}{L} \int \left(\frac{1}{\sqrt{3}} \sin \frac{2\pi x}{L} e^{+4\omega} + \sqrt{\frac{2}{3}} \sin \frac{3\pi x}{L} e^{+9\omega} \right) \left(\frac{1}{\sqrt{3}} \frac{2\pi}{L} \cos \frac{2\pi x}{L} e^{-4\omega} + \sqrt{\frac{2}{3}} \frac{3\pi}{L} \cos \frac{3\pi x}{L} e^{-9\omega} \right) dx$$

$$= \frac{-2i\hbar}{L} \int \Big[\frac{1}{3} \frac{2\pi}{L} \sin \frac{2\pi x}{L} \cos \frac{2\pi x}{L} e^0 + \frac{2}{3} \frac{3\pi}{L} \sin \frac{3\pi x}{L} \cos \frac{3\pi x}{L} e^0$$

(These vanish by $\int \sin ax \cos ax \, dx = 0$ over a cycle)

$$+ \frac{\sqrt{2}}{3} \frac{2\pi}{L} \sin \frac{3\pi x}{L} \cos \frac{2\pi x}{L} e^{5\varepsilon i t/\hbar} + \frac{\sqrt{2}}{3} \frac{3\pi}{L} \sin \frac{2\pi x}{L} \cos \frac{3\pi x}{L} e^{-5\varepsilon i t/\hbar} \Big] dx$$

$$= \frac{-2i\hbar}{L} \left[\frac{\sqrt{2}}{3} \frac{2\pi}{L} I_4 e^{5\varepsilon i t/\hbar} + \frac{\sqrt{2}}{3} \frac{3\pi}{L} I_5 e^{-5\varepsilon i t/\hbar} \right]$$

8.3

QM 2.2 (f) continued

$$\langle p\rangle = -\frac{2i\hbar}{L}\left[\frac{\sqrt{2}}{3}\frac{2\pi}{L} I_4\, e^{5\varepsilon i t/\hbar} + \frac{\sqrt{2}}{3}\frac{3\pi}{L} I_5\, e^{-5\varepsilon i t/\hbar}\right]$$

$$I_4 = \int_0^L \sin\frac{3\pi x}{L}\cos\frac{2\pi x}{L}\,dx \qquad I_5 = \int_0^L \sin\frac{2\pi x}{L}\cos\frac{3\pi x}{L}\,dx$$

$$\int \sin mx \cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}$$ DWIGHT 465 p. 115

$$\int \sin\frac{3\pi x}{L}\cos\frac{2\pi x}{L}\,dx = -\frac{\cos\left(\frac{3\pi}{L}-\frac{2\pi}{L}\right)x}{2\left(\frac{3\pi}{L}-\frac{2\pi}{L}\right)} - \frac{\cos\left(\frac{5\pi}{L}\right)x}{2\cdot\frac{5\pi}{L}} = \frac{-\cos\frac{\pi x}{L}}{\frac{2\pi}{L}} - \frac{\cos\frac{5\pi x}{L}}{\frac{2\pi}{L}\cdot 5}\Bigg|_0^L$$

$$= \frac{L}{2\pi}\left[-(\cos\pi - \cos 0) - \frac{1}{5}(\cos 5\pi - \cos 0)\right] = \frac{L}{2\pi}\left[-(-1-1) - \frac{1}{5}(-1-1)\right]$$

$$= \frac{L}{2\pi}\left[2 + \frac{2}{5}\right] = \frac{L}{\pi}\left[1 + \frac{1}{5}\right] = \underline{\frac{6}{5}\frac{L}{\pi}}$$

$$\int \sin\frac{2\pi x}{L}\cos\frac{3\pi x}{L}\,dx = -\frac{\cos\left(\frac{2\pi}{L}-\frac{3\pi}{L}\right)x}{2\left(-\frac{\pi}{L}\right)} - \frac{\cos\left(\frac{5\pi}{L}\right)}{2\cdot\frac{5\pi}{L}}\Bigg|_0^L = \frac{-L}{2\pi}\left[-\cos\left(\frac{\pi x}{L}\right) + \frac{1}{5}\cos\frac{5\pi x}{L}\right]_0^L$$

$$= -\frac{L}{2\pi}\left[-(\cos\pi - \cos 0) + \frac{1}{5}(\cos 5\pi - \cos 0)\right] = \frac{-L}{2\pi}\left[-(-1-1) + \frac{1}{5}(-1-1)\right]$$

$$= -\frac{L}{2\pi}\left[2 - \frac{2}{5}\right] = -\frac{L}{\pi}\left[1 - \frac{1}{5}\right] = \underline{-\frac{4}{5}\frac{L}{\pi}}$$

$$\langle p\rangle = -\frac{2i\hbar}{L}\frac{\sqrt{2}}{3}\frac{\pi}{L}\left[2\left(\frac{6}{5}\frac{L}{\pi}\right)e^{5\varepsilon i t/\hbar} - 3\left(\frac{4}{5}\frac{L}{\pi}\right)e^{-5\varepsilon i t/\hbar}\right]$$

$$= -\frac{2i\hbar}{L}\frac{\sqrt{2}}{3}\cdot\left[\frac{12}{5}\left(2i\sin\frac{5\varepsilon t}{\hbar}\right)\right]$$

$$= \frac{4\hbar}{L}\sqrt{2}\,\frac{4}{5}\sin\frac{5\varepsilon t}{\hbar} = \underline{\frac{16\sqrt{2}}{5}\frac{\hbar}{L}\sin\frac{5\varepsilon t}{\hbar}}$$

(g) $\boxed{\omega = \frac{5\varepsilon}{\hbar}}$

8.4

QM 3. **Application:** Imagine that an electron is confined in a one-dimensional box with a width of 3Å (roughly the size of an atom). (13 pts)

$L =$

$\hbar c = 1973\ eV \cdot \mathring{A}$

$m_e c^2 = .511\ MeV$

$hc = 1240\ eV \cdot nm$

(3) (a) Calculate the three lowest allowed values of the electron energy. Express your answers in eV.

(3) (b) Suppose that the electron can be excited from a lower to a higher energy state by absorption of an incident photon (as in an atom). What would be the wavelength of light needed to excite it from the ground state to the highest energy level computed in (a)?

(3) (c) What would be the possible wavelengths of light that could be emitted by the electron following the excitation described in (b)?

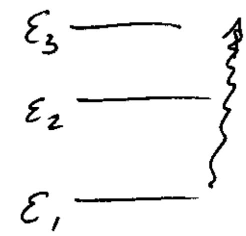
(3) (d) Draw an energy level diagram including the energies and transition wavelengths, to summarize your results.

(+1) free (e) *Optional challenge question: Describe how this would change in the presence of an external magnetic field.*

(a) $E_n = n^2 \mathcal{E}$ where $\mathcal{E} = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (\hbar c)^2}{2mc^2 L^2} \simeq \frac{10}{2(\frac{1}{2}\times 10^6 eV)} \left(\frac{2000\ eV \cdot \mathring{A}}{3\mathring{A}}\right)^2$

$$\mathcal{E} \simeq 10^{-5} \times \frac{4\times 10^6}{10}\ eV \sim 4eV$$

$\mathcal{E}_1 \simeq 4eV$ $\quad \mathcal{E}_2 = 4\mathcal{E}_1 \simeq 16\ eV$ $\quad \mathcal{E}_3 = 9\mathcal{E}_1 \simeq 36\ eV$



(b) $\Delta E = \frac{hc}{\lambda}$ $\quad \boxed{\lambda_{31} = \frac{hc}{\mathcal{E}_3 - \mathcal{E}_1}} = \frac{hc}{(9-1)\mathcal{E}_1} = \frac{hc}{8\mathcal{E}_1}$

$\lambda_{31} = \frac{1240\ eV\cdot nm}{32\ eV} \sim 40\ nm$ - very short wavelength EUV

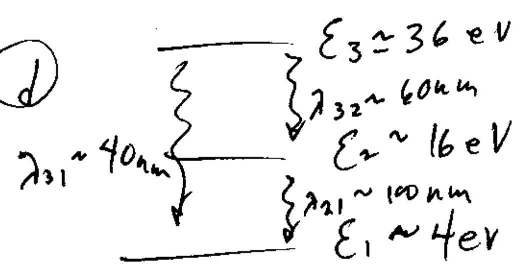
(c)

Electron in $\mathcal{E}_3$ could emit light of wavelength λ_{31}

or $\lambda_{32} = \frac{hc}{\mathcal{E}_3 - \mathcal{E}_2} = \frac{1240\ eV\cdot nm}{(9-4)\mathcal{E}_1\ eV} = \frac{1240}{5.4}\ nm \simeq 60\ nm$ (EUV)

Could also relax from $\mathcal{E}_2$ to $\mathcal{E}_1$: $\lambda_{21} = \frac{hc}{\mathcal{E}_2 - \mathcal{E}_1} = \frac{1240\ eV\cdot nm}{(4-1)\cdot 4eV} = \frac{1240}{3.4} \sim 100\ nm$

(d)



Moore SP5.1

QM 4: **Short answer questions:** (20 pts) (Moore's Last Exam)

(6) (a) Is it possible to find a basis in which the matrices A and B below are both diagonal? If so, show how.

$A = 2I$ ← identity operator

$$A \leftrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad B \leftrightarrow \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

yes, if they commute. I commutes with all operators $\therefore [A,B]=0$ $\therefore$ simultaneously diagonalizable.

(b) What would it mean physically if Hermitian A and B were simultaneously diagonalizable?

If they are Hermitian, their operators correspond to measurements of observables. If they commute, their observables are simultaneously measurable (like L_z and L^2, but not x and p).

They share an eigenbasis, so an observation of one will not interfere with subsequent measurements of the other.

(2) (c) If an operator C commutes with the Hamiltonian H of a system, then <C> is conserved in time for that system. Explain what that means, and why it's so.

By Ehrenfest's theorem, $\frac{d}{dt}\langle C \rangle = \frac{-i}{\hbar}\langle [C,H] \rangle = 0$ if $[C,H]=0$. Therefore $\langle C \rangle$ is conserved in time. That is, the expectation value of the observable C does not change.

Bjørn found diagonalizing matrix $U_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, $U_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Summer

$$U_B^{+} B U_B = \beta \qquad U_A^{+} A U_A = \alpha = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $U_1 U_2 = I U_2 = U_2$ diagonalizes both $A + B$:

$$U_2^{+} B U_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$
$$U_2^{+} A U_2 = U_2^{+} 2I U_2 = 2I U_2^{+} U = 2I \checkmark$$

(QM 4: **Short answer questions** continued …)

(4) (d) Consider **three indistinguishable particles** of mass m in a one-dimensional box potential of width a. If the total energy of the system is $E_{tot} = \frac{11\hbar^2\pi^2}{2ma^2}$, are the particles bosons, fermions, or can't one tell?

The total energy of the system is the sum of the energies of the individual particles. Since $\mathcal{E} = \frac{\hbar^2\pi^2}{2ma^2}$ is the energy of one particle in the ground state of the box, and $E_n = n^2\mathcal{E}$ for higher states,

$E_{tot} = E_{n_1} + E_{n_2} + E_{n_3} = \mathcal{E}(n_1^2 + n_2^2 + n_3^2) = 11\,\mathcal{E} \rightarrow \underline{n_1^2 + n_2^2 + n_3^2 = 11}$

The only set of integers whose sums square to 11 are $1^2 + 1^2 + 3^2 = 11$

$n_1 = 1,\ n_2 = 1,\ n_3 = 3$. We cannot have $n_1 = n_2$ for fermions. $\boxed{\text{THESE MUST BE BOSONS}}$

(8) (e) The state of a particle in the position basis is given by $\psi(r) = Ae^{-\alpha r^2}\sin\theta\cos\phi$.
(i) Expand the state $\psi(r)$ in the angular momentum eigenstates (hint: use Euler's relation).
(ii) What are the quantum numbers ℓ and m?
(iii) What are the possible results if we measure L_z? If we measure L^2?

(a) $\cos\varphi = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$

So $\psi(r) = Ae^{-\alpha r^2}\sin\theta\,\frac{1}{2}(e^{i\varphi} + e^{-i\varphi}) = \frac{1}{2}Ae^{-\alpha r^2}\sin\theta e^{i\varphi} + \frac{1}{2}Ae^{-\alpha r^2}\sin\theta e^{-i\varphi}$

Angular momentum eigenstates: $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$ $\frac{(12.5.39)}{346}$

$\psi(r) = \frac{1}{2}Ae^{-\alpha r^2}\left(-\sqrt{\frac{8\pi}{3}}\,Y_1^{+1}(\theta,\varphi)\right) + \frac{1}{2}Ae^{-\alpha r^2}\sqrt{\frac{8\pi}{3}}\,Y_1^{-1}(\theta,\varphi)$

$= -R(r)\,Y_1^{+1}(\theta,\varphi) + R(r)\,Y_1^{-1}(\theta,\varphi)$

(b) $\underline{\ell = 1 \quad m = \pm 1}$

(c) $L_z\psi = m\hbar\psi \rightarrow \underline{L_z = \pm\hbar}$ $\frac{(12.5.18)}{335}$

$L^2\psi = \ell(\ell+1)\hbar^2\psi \rightarrow \underline{L^2 = 1\cdot 2\hbar^2 = 2\hbar^2}$