The Geometry of Space: Triangles and Coordinate Systems

Take some time to draw various triangles. Draw small ones and large ones, odd ones and regular ones. Are there any simple relationships about the sides of a triangle that you can find?

Now draw just two sides of a triangle. Once you've done that, isn't the third side already determined? So once you know the length of two sides of a triangle, do you have all the information you need to find the length of the third side?

This is a slight trick question. Clearly it is not enough to know just the length of the two sides, you must also know the directions of the two sides. Or equivalently, you need to know the angle between the two sides.

1 Right triangles

Apart from their utility in the physical world, about which we will talk later, right triangles are unique because, by definition, you already know the angle between two sides of a right triangle. So, for a right triangle, it is enough to know the length of those sides to determine the length of the third side.

As it turns out, the calculation needed to determine this is very simple. It is known as the Pythagorean Theorem, and it states that if a and b are the lengths of the sides which form a right angle and c is the length of the third (and longest) side of a triangle, the following relationship holds:

$$c^2 = a^2 + b^2$$

1.1 Experimental verification

Before going any further, you should immediately attempt to verify the Pythagorean Theorem by testing it in the real world with real triangles. To do this, you should find some right triangles and measure the lengths of all three sides. You should also find some non-right triangles and measure the lengths of all three sides. Use your measurements to check the Pythagorean Theorem.

1.2 Solving problems

Once you have satisfied yourself that the Pythagorean Theorem is at least close to the truth, you should take some time to practice working some problems which use the Pythagorean Theorem. In other words, make up some situations in which you know the lengths of two sides of a right triangle, and use that information to find the length of the third side. There are basically two possible scenarios here. Either you know a and b and are trying to find c or you know c and either a or b and you are trying to find the length of the third side.

2 Coordinate systems

What does it mean to say that we live in a three-dimensional space? What does it mean if something is only two-dimensional or one-dimensional?

One way to understand this is to think about addressing a location – that is, assigning a number to a location. In a one-dimensional space (a line or curve), you can specify any location (along the line) by using only one number. In a two-dimensional space (a plane or sheet), you would not be able to use only one number to differentiate all the possible locations. You would need at least two numbers. Similarly, in a three dimensional space, at least three numbers must be used to fully specify a location.

2.1 Conventional notation

There are many, many ways we could come up with to assign numbers to a region of space. Just for convenience, so we can communicate easily and effectively with each other, we have to come to some agreement about how to set up an addressing system and how to write the addresses. One such "convention" for writing addresses is to use a Cartesian coordinate system in which the axes of the coordinates are laid out at right angles to each other. One axis is said to point in the x direction. Another axis, perpendicular to the first, is said to point in the y direction. And a third axis, if needed, is set perpendicular to each of the first two and said to point in the z direction.

Even if you and I agree to use this type of coordinate system, we will still have to take specific steps to ensure that we assign the same numbers to specific locations in space. We must use the same origin and the same scale for our axes, and we have to make sure the axes point in the same directions.

But how do we decide which way to point the axes? How do we decide where to place the origin? How do we decide what scale to use?

It doesn't matter!

And the very fact that it doesn't matter – that it can't matter – is an interesting thing about the world itself. The numbers we define as an address for some point in space are arbitrary! The address is not determined by the world, it is determined by us – by our conventions.

One last piece of convention, again to help us communicate with each other, is that we write addresses in a Cartesian system in a certain order. We write the numbers for the x, y, and z axes in alphabetical order, and we usually put parentheses around the full address. So, the address

(1 cm, 3 cm, -5 cm)

means, by convention, one centimeter from the origin in the positive direction along the x axis, three centimeters from the origin in the positive direction along the y axis, and five centimeters from the origin in the negative direction along the z axis. Notice how much shorter it is to use the notation (1 cm, 3 cm, -5 cm) than to write it out in words. In fact, if you know you are using centimeters, you can just write (1, 3, -5). But you have to know the convention to be able to interpret the address in its shortened form.