Non-Linear Dynamics Homework Solutions
Week 3

January 25, 2009

4.1.2 Find and classify all of the fixed points of

$$\dot{\theta} = 1 + 2\cos\theta$$

and sketch the phase portrait on the circle.

Fixed points occur at values θ^* such that $0 = 1 + 2\cos\theta^*$. Since this system is periodic with period 2π, we only need to consider values of θ^* in between 0 and 2π. Throughout one full period, $2\cos\theta$ varies continuously from 2 to -2 and then back to 2 at the end of the period. Thus, once on the way to -2 and once coming from -2 back up, $\cos\theta = -1$ will be true and consequently, $\dot{\theta} = 0$. Since $\dot{\theta} = -1$ when $\theta = \pi$, We must have the first fixed point before that value and the second one after. We also know that when $\theta = \pi/2$ or $3\pi/2$, $\dot{\theta} = 1$, so we let θ_1^* be the fixed point lying in the second quadrant and θ_2^* be the fixed point lying in the third.

For $\theta > 0$ but less that θ_1^*, we note that $\dot{\theta}$ is positive. For θ between θ_1^* and θ_2^*, $\dot{\theta}$ is negative. This means that θ_1^* is a stable fixed point. From what we said above, along with the observation that for $\theta > \theta_2^*$ but less than 2π, $\dot{\theta} > 0$, we have that θ_2^* is an unstable fixed point. Piecing this information together we get the following phase diagram on the circle.

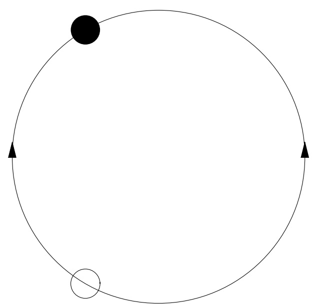

Figure 1: Vector Field on a Circle

4.3.3 We consider the system

$$\dot{\theta} = \mu\sin\theta - \sin 2\theta.$$

To find the fixed points of the system we set $\dot{\theta} = 0$ and find that $\mu \sin\theta = 2\sin\theta\cos\theta$ by the double angle formula. Then either $\sin\theta = 0$ implying that $\theta^* = 0$ or π, or we can divide by $\sin\theta$, in which case we must have that $\mu/2 = \cos\theta^*$. It should now be apparent that for different values of μ will have different numbers of fixed points. Figure 2 gives a 3-D plot of $\dot{\theta}$ as a function of μ and θ.

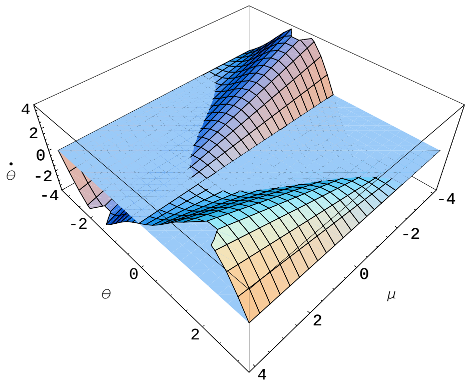

Figure 2: 3-D Plot of $\dot{theta}$ vs. θ and μ

From this we can extract a bifurcation diagram by considering it's intersection with the $\dot{\theta} = 0$ plane. The curves of intersection will form the shape of the bifurcation diagram and we can get stability information by considering the sign of $\dot{\theta}$ on each side of the bifurcation curve. Doing all this we get the bifurcation diagram of Figure 3 and can see that we have pitchfork bifurcations at the critical values of $\mu_c = \pm 2$. Note that we can also obtain these μ_c values by noting that solitary solutions to the equation $\mu/2 = \cos\theta^*$ first appear at these points and vanish, that two solutions are present for intermediate values, and that no solution exist for μ values outside of the interval.

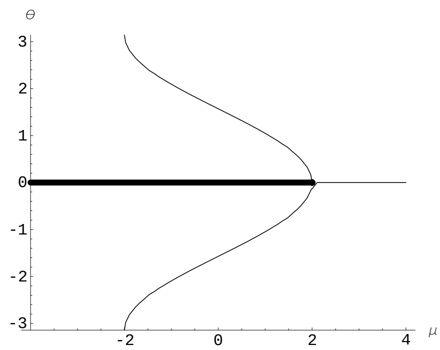

Figure 3: Bifurcation Diagram

Using all of this information we get the phase portraits of Figure 4 as we vary μ.

4.4.4 *Torsion Spring* We study the equation of motion

$$b\dot{\theta} + mgL\sin\theta = \Gamma - k\theta$$

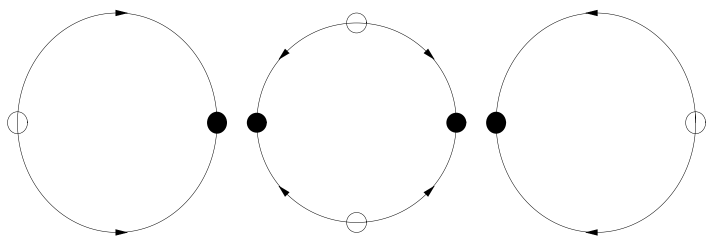

Figure 4: Vector Field as μ is Varied From Below -2 to Above 2

a) Does this system correspond to a well defined vector field?
Not unless $k = 0$. This becomes particularly apparent when noting that

$$\dot{\theta} = \frac{\Gamma - mgL\sin\theta}{b} - \frac{-k}{b}\theta.$$

The $k\theta$ term is not periodic but the term on the left of the sum is with period 2π. From this fact the system is periodic iff $k = .0$

b) We nondimensionalize the system.
We want to define our dimensionless units so that things simplify. With this in mind, we define $t = b/(mgL)\tau$. Then $dt/d\tau = b/mgL$ so that by the chain rule

$$\frac{d\theta}{d\tau} = \dot{\theta}\frac{dt}{d\tau} = \frac{\Gamma}{mgL} - \sin\theta - \frac{k}{mgL}\theta.$$

We shall further simplify this by letting $c = k/(mgL)$ and $\Gamma' = \Gamma/(gmL)$ So that we now have the equation

$$\frac{d\theta}{d\tau} = \Gamma' - c\theta - \sin\theta.$$

This checks out as being dimensionless. Since $\sin\theta$ is a summand, and is dimensionless, the other summands must be as well.

c) The system must be stable overall. You won't end up catching a ride to infinity from any starting point in this sytem, since we assume that $k \geq 0$. Under the assumption of $k > 0$, the value of the line $\Gamma' - c\theta$, will be much greater than $-\sin\theta$ for all θ less than some sufficiently negative value. The idea is supported by the illustrations of Figure 7.

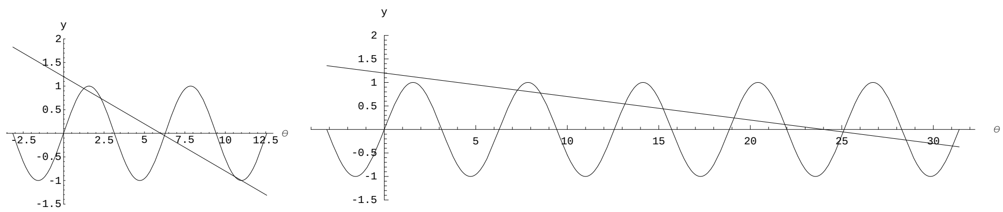

Figure 5: Intersections

d) Figure 7 displays these intersections for the same Γ' values but different c values. It becomes clear from considering the diagrams that as the slope of the line approaches zero from below, infinitely many intersections arise, and the nature of these intersections implies that we have *blue-sky bifurcations* arising at these points.