15.30 $\star$ If we define ϕ so that $\cosh\phi = \gamma$, then

$$\sinh\phi = \sqrt{\cosh^2\phi - 1} = \sqrt{\gamma^2 - 1} = \sqrt{\frac{1}{1-\beta^2} - 1} = \sqrt{\frac{\beta^2}{1-\beta^2}} = \frac{\beta}{\sqrt{1-\beta^2}} = \gamma\beta,$$

and $\tanh\phi = (\sinh\phi)/(\cosh\phi) = \gamma\beta/\beta = \beta$.

15.31 $\star$ The speed $v = \beta c$ of C relative to A is given by the inverse velocity addition formula:

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + (\tanh\phi_1)(\tanh\phi_2)} = \tanh(\phi_1 + \phi_2)$$

(In the last step I used the "well known" addition formula for the hyperbolic tangent. One way to prove this is to start from the corresponding formula for the more familiar trigonometric tangent.) That is, the velocity of C relative to A is given by $\beta = \tanh\phi$ where the rapidity ϕ is just $\phi = \phi_1 + \phi_2$.

15.37 $\star$ If x and y are four-vectors, then under the standard boost $x_1' = \gamma(x_1 - \beta x_4)$, $x_2' = x_2$, and so on. Therefore

$$\begin{aligned} x'\cdot y' &= x_1'y_1' + x_2'y_2' + x_3'y_3' - x_4'y_4' \\ &= \gamma(x_1 - \beta x_4)\gamma(y_1 - \beta y_4) + x_2y_2 + x_3y_3 - \gamma(x_4 - \beta x_1)\gamma(y_4 - \beta y_1) \\ &= \gamma^2(1-\beta^2)x_1y_1 + x_2y_2 + x_3y_3 - \gamma^2(1-\beta^2)x_4y_4 \end{aligned}$$

since the cross terms involving x_1y_4 and x_4y_1 in the second line all cancelled. Finally, notice that $\gamma^2(1-\beta^2) = 1$, so the last line is just $x\cdot y$, and we've proved that $x'\cdot y' = x\cdot y$.

15.43 $\star$ **(a)** Suppose that the body moves from $\mathbf{x}$ to $\mathbf{x} + d\mathbf{x}$ as the time advances from t to $t + dt$. Let dx denote the four-vector displacement $dx = (d\mathbf{x}, c\,dt) = (\mathbf{v}, c)dt$ and consider the following two equivalent statments:

$$|\mathbf{v}| < c \iff dx^2 < 0. \tag{ix}$$

Since dx^2 is Lorentz invariant, the second condition, if true in one frame, must be true in all frames. The same must therefore apply to the first; that is, if $|\mathbf{v}| < c$ in one frame, then $|\mathbf{v}| < c$ in all frames.

(b) The argument for a signal with speed c is the same except that the two conditions (ix) are replaced by $|\mathbf{v}| = c \iff dx^2 = 0$.

15.47 ⋆ Since he had to be approaching the light head-on (or nearly so), $\theta = 0$ and Eq.(15.64) becomes $\omega = \omega_o\sqrt{(1+\beta)/(1-\beta)}$ (as in Problem 1.46). Solving for β we find that

$$\beta = \frac{\omega^2 - \omega_o^2}{\omega^2 + \omega_o^2} = \frac{\lambda_o^2 - \lambda^2}{\lambda_o^2 + \lambda^2} = \frac{65^2 - 53^2}{65^2 + 53^2} = 0.20.$$

His speed had to be about $0.20c$.

15.48 ⋆⋆ **(a)** With $\theta = 90°$, Eq.(15.64) reduces to $\omega = \omega_o/\gamma$. If $\beta = 0.2$, then $1/\gamma = \sqrt{1-\beta^2} = \sqrt{1-0.04} \approx 1 - 0.02$. Therefore the percent shift is -2%.

(b) If the source approaches head-on, the observed frequency is $\omega = \omega_o/\gamma(1-\beta) \approx \omega_o(1-0.02)/(1-0.2) \approx 1.22\omega_o$, and the percent shift is $+22\%$.
