4.1

6) $u = e^x \cos y$

a) WTS: $\partial^2 u/\partial x \partial y = \partial^2 u/\partial y \partial x$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -e^x \sin y \qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial y \partial x} = -e^x \sin y \qquad \frac{\partial^2 u}{\partial x \partial y} = -e^x \sin y$$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

b) WTS: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(e^x \cos y) \qquad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(-e^x \sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

4.4

7) $F = \frac{k q_1 q_2}{r^2}$ Find relative error in q_2 if relative error in F is 2%; q_1, 3%; r, 5%.

$$dF = \pm .02 F \Rightarrow \left|\frac{dF}{F}\right| = .02$$

$$dq_1 = \pm .03 q_1 \Rightarrow \left|\frac{dq_1}{q_1}\right| = .03$$

$$dr = \pm .05 r \Rightarrow \left|\frac{dr}{r}\right| = .05$$

$q_2 = \frac{Fr^2}{kq_1}$ taking $\ln$ of both sides we get

$$\ln q_2 = \ln \frac{Fr^2}{kq_1}$$

$$\ln q_2 = \ln F + 2\ln r - \ln k - \ln q_1$$

taking the derivative

$$\left|\frac{dq_2}{q_2}\right| = \left|\frac{dF}{F} + \frac{2dr}{r} - \frac{dq_1}{q_1}\right| \le \left|\frac{dF}{F}\right| + 2\left|\frac{dr}{r}\right| + \left|\frac{dq_1}{q_1}\right|$$

$$\le .02 + 2(.05) + .03$$

$$\le .15 = 15\%$$

4.5

3) $r = e^{-p^2-q^2}$, $p = e^s$, $q = e^{-s}$

WTF: $\frac{dr}{ds}$

$$\frac{dr}{ds} = \frac{\partial r}{\partial p}\frac{dp}{ds} + \frac{\partial r}{\partial q}\frac{dq}{ds}$$
$$= -2pr \cdot p - 2qr(-q)$$
$$= -2p^2r + 2q^2r$$
$$= 2r(q^2-p^2)$$

4.7

22) $w = f(ax+by)$

WTS: $b\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} = 0$

let $z = ax+by$, then $w = f(z)$ and

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} = a\frac{dw}{dz}$$
$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y} = b\frac{dw}{dz}$$

so

$$b\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y} = ba\frac{dw}{dz} - ab\frac{dw}{dz} = 0$$

Q.E.D.B.