4. **(a)** The equilibrium points occur where the vector field is zero, that is, at solutions of

$$\begin{cases} -x = 0 \\ -4x^3 + y = 0. \end{cases}$$

So, $x = y = 0$ is the only equilibrium point.

(b) The Jacobian matrix of this system is

$$\begin{pmatrix} -1 & 0 \\ -12x^2 & 1 \end{pmatrix},$$

which at (0, 0) is equal to

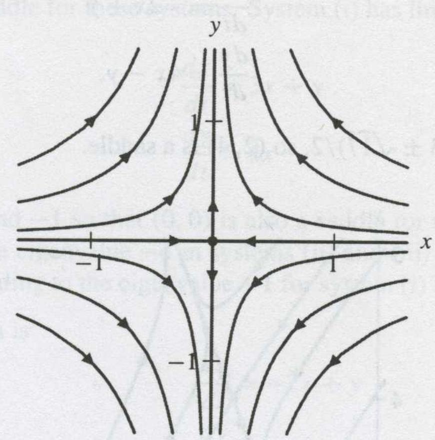
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So the linearized system at (0, 0) is

$$\frac{dx}{dt} = -x$$
$$\frac{dy}{dt} = y$$

(we could also see this by "dropping the higher order terms").

(c) The eigenvalues of the linearized system at the origin are -1 and 1, so the origin is a sadd
The linearized system decouples, so solutions approach the origin along the x-axis and te
away form the origin along the y-axis.



5. **(a)** Using separation of variables (or simple guessing), we have $x(t) = x_0 e^{-t}$.

(b) The equation

$$\frac{dy}{dt} = -4x^3 + y$$

o, that is, at solutions of

and 1, so the origin is a saddle.
igin along the x-axis and tend

e^{-t}.

is a first-order, linear equation. We write the equation as

$$\frac{dy}{dt} - y = -4x^3.$$

Therefore, the integrating factor is e^{-t}. Multiplying both sides of the equation by e^{-t} yields

$$\left(\frac{dy}{dt} - y\right) e^{-t} = -4x^3 e^{-t}.$$

Note that the left-hand side is just the derivative of ye^{-t}, and the right-hand side is $-4x_0^3 e^{-3t} e$
since $x(t) = x_0 e^{-t}$. Therefore we have

$$\frac{d}{dt} ye^{-t} = -4x_0^3 e^{-3t} e^{-t} = -4x_0^3 e^{-4t}.$$

After integrating and simplifying, we have

$$y(t) = x_0^3 e^{-3t} + (y_0 - x_0^3)e^t.$$

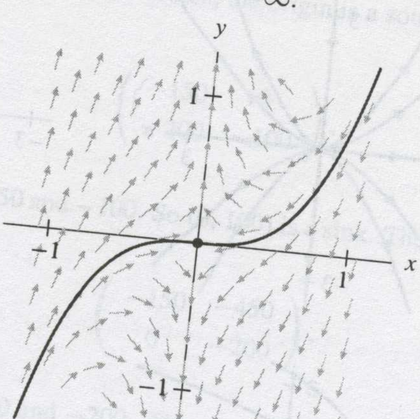
(c) The general solution of the system is

$$x(t) = x_0 e^{-t}$$
$$y(t) = x_0^3 e^{-3t} + (y_0 - x_0^3)e^t.$$

(d) For all solutions, $x(t) \to 0$ as $t \to \infty$. For a solution to tend to the origin as $t \to \infty$, we must have $y(t) \to 0$, and this can happen only if $y_0 - x_0^3 = 0$.

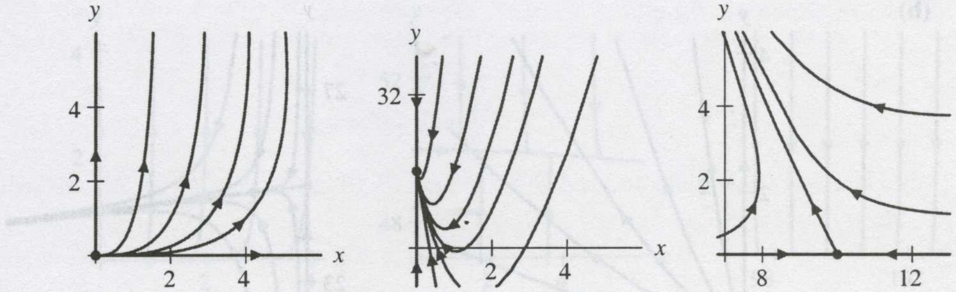
(e) Since $x = x_0 e^{-t}$, we see that a solution will tend toward the origin as $t \to -\infty$ only if $x_0 = 0$. In that case, $y(t) = y_0 e^t$, and $y(t) \to 0$ as $t \to -\infty$.

(f)



(g) Solutions tend away from the origin along the y-axis in both systems. In the nonlinear system, solutions approach the origin along the curve $y = x^3$ which is tangent to the x-axis. For the linearized system, solutions tend to the origin along the x-axis. Near the origin, the phase portraits are almost the same.

(b)



9. **(a)** The equilibrium points are (0, 0), (0, 25), (100, 0) and (75, 12.5). We classify these equilibrium points by computing the Jacobian matrix, which is

$$\begin{pmatrix} 100 - 2x - 2y & -2x \\ -y & 150 - x - 12y \end{pmatrix},$$

and evaluating it at each of the equilibrium points. At (0, 0), the Jacobian matrix is

$$\begin{pmatrix} 100 & 0 \\ 0 & 150 \end{pmatrix},$$

and the eigenvalues are 100 and 150. So this point is a source. At (0, 25), the Jacobian matrix is

$$\begin{pmatrix} 50 & 0 \\ -25 & -150 \end{pmatrix},$$

and the eigenvalues are 50 and −150. Hence, this point is a saddle. At (100, 0), the Jacobian matrix is

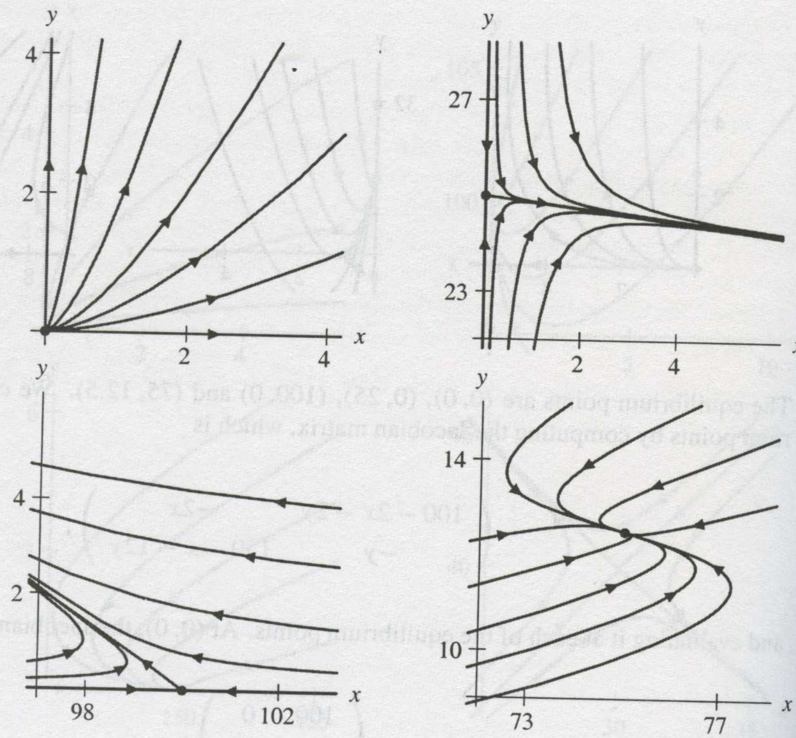
$$\begin{pmatrix} -100 & -200 \\ 0 & 50 \end{pmatrix},$$

and the eigenvalues are −100 and 50. Therefore, this point is a saddle. Finally, at (75, 12.5), the Jacobian matrix is

$$\begin{pmatrix} -75 & -150 \\ -12.5 & -75 \end{pmatrix}$$

and the eigenvalues are approximately −32 and −118. So this point is a sink.

(b)



10. **(a)** The equilibrium points in the first quadrant are (0, 0), (0, 50) and (100, 0). To classify these equilibrium points, we compute the Jacobian matrix, which is

$$\begin{pmatrix} -2x - y + 100 & -x \\ -2xy & -x^2 - 3y^2 + 2500 \end{pmatrix},$$

and evaluate it at each point. At (0, 0), the Jacobian matrix is

$$\begin{pmatrix} 100 & 0 \\ 0 & 2500 \end{pmatrix},$$

which has eigenvalues 100 and 2500. So (0, 0) is a source. At (0, 50), the Jacobian matrix is

$$\begin{pmatrix} 50 & 0 \\ 0 & -5000 \end{pmatrix},$$

which has eigenvalues -10 and -5000. Hence, (0, 50) is a saddle. At (100, 0), the Jacobian matrix is

$$\begin{pmatrix} -100 & -100 \\ 0 & 900 \end{pmatrix},$$

which has eigenvalues -40 and -7500. Thus, (100, 0) is a sink.

(f) The reason the linearizations and the nonlinear system look so different is that the equation for dx/dt contains only higher-order terms (just x^3 in this case). Since the equilibrium points occur along the y-axis ($x = 0$), the linearization has an entire line of equilibria in the x-direction.

18. **(a)** The equation $x^2 - a = 0$ has no solutions if $a < 0$.

(b) The equilibrium points are $(\pm\sqrt{a}, 0)$.

(c) When $a = 0$, the only equilibrium point is $(0, 0)$.

(d) The Jacobian matrix is

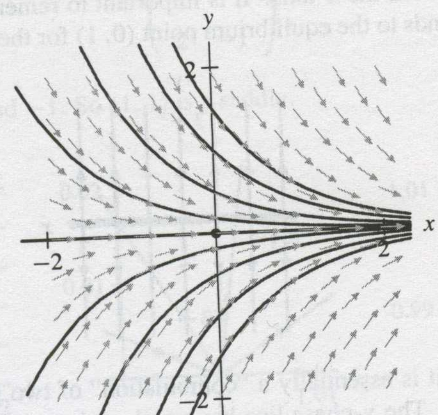
$$\begin{pmatrix} 2x & 0 \\ -2xy & -x^2 - 1 \end{pmatrix}.$$

At $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which has eigenvalues -1 and 0. So $(0, 0)$ is a node.

19. **(a)**



(b) The linearization of the equilibrium point at the origin has the coefficient matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which has eigenvalues -1 and 0. So for the linearized system, the x-axis is a line of equilibria and solutions tend to zero in the y-direction. The nonlinear terms make solutions tend to zero in the x-direction for initial conditions with $x < 0$ and away from zero in the x-direction for initial conditions with $x > 0$.

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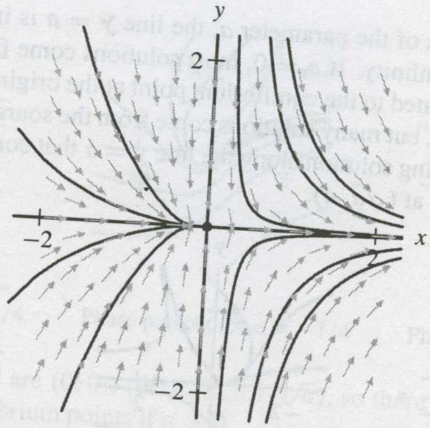


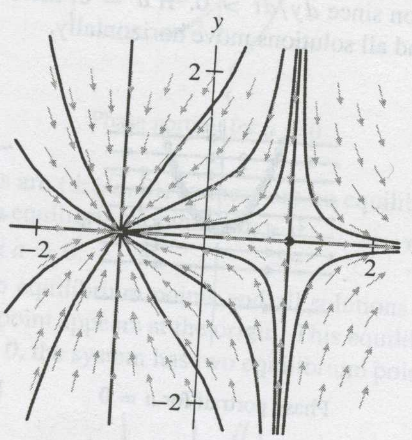
(c) The equilibria are $(\pm 1, 0)$. The coefficient matrix of the linearization at $(1, 0)$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The eigenvalues are 2 and -2, thus $(1, 0)$ is a saddle. The coefficient matrix of the linearizat at $(-1, 0)$ is

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

which has -2 as a repeated eigenvalue. So, $(-1, 0)$ is a sink.



20. **(a)** The equilibrium points are $(\pm\sqrt{a}, a)$, so there are no equilibrium points if $a < 0$, one equilibrium point if $a = 0$, and two equilibrium points if $a > 0$

(b) If $a = 0$, the equilibrium point at the origin has eigenvalues 0 and 1 and is a node. If $a > 0$, the system has two equilibrium points, a saddle at $(\sqrt{a}, a)$ with eigenvalues $-2\sqrt{a}$ and 1 and a source at $(-\sqrt{a}, a)$ with eigenvalues $2\sqrt{a}$ and 1. A bifurcation occurs at $a = 0$ because the number of equilibrium points changes. It also reasonable to say that there is a bifurcation at $a = 1/4$ because the source at $(-\sqrt{a}, a)$ has repeated eigenvalues. For all other positive values of a, these eigenvalues are real and distinct.