Hartle Week 4 Solutions Taiyo Terada

5.6 (a) No, notice $(gt)^2 < 1+g^2t^2$

$$\Rightarrow gt < \sqrt{1+g^2t^2} \Rightarrow \frac{gt}{\sqrt{1+g^2t^2}} < 1.$$

but as $t \to \infty$, $gt/\sqrt{1+g^2t^2} \to 1 = c$ (speed of light)

(b) $$\underset{\sim}{u} = \left(\frac{dt}{d\tau}, \frac{dx}{dt}\frac{dt}{d\tau}\right) = \left(\frac{1}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}}, \frac{1}{\sqrt{1-\left(\frac{dx}{dt}\right)^2}}\frac{gt}{\sqrt{1+g^2t^2}}\right)$$

note: $$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{(gt)^2}{1+g^2t^2}}} = \left(\frac{1+g^2t^2-g^2t^2}{1+g^2t^2}\right)^{-1/2}$$

$$= \left(\frac{1}{1+g^2t^2}\right)^{-1/2} = \sqrt{1+g^2t^2}$$

So $$\underset{\sim}{u} = \left(\sqrt{1+g^2t^2}, gt\right) = \left(\cosh(g\tau), \sinh(g\tau)\right)$$ as we will show

(c) $$\frac{dt}{d\tau} = \sqrt{1+g^2t^2} \Rightarrow \int \frac{dt}{\sqrt{1+g^2t^2}} = \int d\tau$$

let $t = \frac{\sinh(\theta)}{g}$, $\theta = \operatorname{arcsinh}(gt)$ $$\Rightarrow \frac{1}{g}\int \frac{\cosh\theta\, d\theta}{\sqrt{1+\sinh^2\theta}} = \tau$$

$$\Rightarrow \frac{\theta}{g} = \tau \Rightarrow \boxed{gt = \sinh(g\tau)}$$

and hence $\frac{dt}{d\tau} = \cosh(g\tau)$

and $$\frac{dx}{dt}\frac{dt}{d\tau} = \frac{dx}{d\tau} = gt \Rightarrow \boxed{x = \frac{\cosh(g\tau)}{g}}$$

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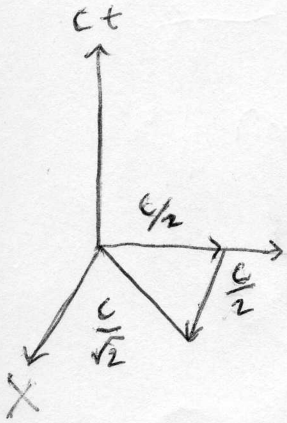
$\boxed{5.6}$ continued (d) $\underset{\sim}{f} = m\underset{\sim}{a} = m\frac{d\underset{\sim}{u}}{d\tau} = (\gamma \vec{F}\cdot\vec{V}, \gamma\vec{F})$

$\underset{\sim}{u} = (\cosh(g\tau), \sinh(g\tau))$ from Part (c) γ

$$\Rightarrow \boxed{\underset{\sim}{f} = m(g\sinh(g\tau), g\cosh(g\tau))} \Rightarrow \gamma\vec{F} = mg\cosh(g\tau)$$

$$\Rightarrow \boxed{\vec{F} = mg}$$

$\boxed{5.8}$ $\underset{\sim}{u} = (\gamma, \gamma\vec{V}) = \left(\frac{1}{\sqrt{1-\frac{1}{2}}}, \frac{1}{\sqrt{1-\frac{1}{2}}}\vec{V}\right) = (\sqrt{2}, \sqrt{2}\vec{V})$



(a) $\vec{V} = \left(\frac{c}{2}, \frac{c}{2}, 0\right) \Rightarrow \boxed{\underset{\sim}{u} = \left(\sqrt{2}, \frac{\sqrt{2}c}{2}, \frac{\sqrt{2}c}{2}, 0\right)}$

(b) $\underset{\sim}{p} = (\gamma mc^2, \gamma m\vec{v}) = \left(\sqrt{2}mc^2, \sqrt{2}m\left(\frac{c}{2}, \frac{c}{2}, 0\right)\right)$

$$\boxed{= \left(\sqrt{2}mc^2, \frac{\sqrt{2}}{2}mc, \frac{\sqrt{2}}{2}mc, 0\right)}$$

for actual values with mass $135 M_e = M$

$$\underset{\sim}{p} = \left(\sqrt{2}M, \frac{\sqrt{2}}{2}M, \frac{\sqrt{2}}{2}M, 0\right)$$

$\boxed{5.9}$ $E = \gamma m_e = 40\,\text{GeV} = 40\cdot10^3\,\text{MeV}$ ← conversion from eV to g.

$m_e = 9.11\times10^{-28}\,g\left(\frac{931.5\,\text{MeV}}{1.6\times10^{-24}\,g}\right)$, assuming $c=1$

$$\Rightarrow \gamma = \frac{40\cdot10^3\,\text{MeV}}{m_e\,\text{MeV}} = \frac{40\cdot10^3}{.53} = 75471.7$$

$\Rightarrow$ by lorentz contraction the pipe length $l = 2\,\text{mi} = 3218.68\,\text{m}$

$$l = l'\cdot\gamma \Rightarrow l' = l/\gamma = \frac{3218.68}{75471.7} \approx .0426\,\text{m}$$

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$\boxed{5.12}$ (a) $\frac{d\vec{p}}{dt} = e\vec{E}$, constant charge e and electric field $\vec{E}$ means constant force, so $p^1 = F^1 t$ along x direction (superscript $1 = x$)

$\underline{p} = (\gamma m, \gamma m\vec{v}) \Rightarrow \vec{p} = \gamma m\vec{v} = \gamma m v^1$.

So $\gamma m v^1 = F^1 t = e|\vec{E}|t \Rightarrow v^1 = \frac{F^1 t}{\gamma m} = \frac{e|\vec{E}|t}{\gamma m}$

[I'll leave off the "1" from now on]

$\Rightarrow \frac{dx}{dt} = \frac{Ft}{m\sqrt{1+\left(\frac{Ft}{m}\right)^2}}$, see Page 650 example 15.11 in Taylor. or use equation 5.47 in Hartle

$\Rightarrow \int dx = \int \frac{Ft}{m\sqrt{1+\left(Ft/m\right)^2}}dt$, the substitution $u = 1+\left(\frac{Ft}{m}\right)^2$ reveals $\left[du = \frac{2F^2}{m^2}t\right]$

$\Rightarrow X = \frac{m}{2F}\int \frac{du}{\sqrt{u}} = \frac{m}{2F}2\sqrt{u} = \frac{m}{F}\sqrt{1+\left(Ft/m\right)^2} + C$

If the partical starts at the begining of the accelerater $X=0, t=0$

$\Rightarrow 0 = \frac{m}{F} + C \Rightarrow C = -\frac{m}{F} \Rightarrow \boxed{X = \frac{m}{F}\sqrt{1+\left(\frac{Ft}{m}\right)^2} - \frac{m}{F},\ F = e|\vec{E}|}$

(b) So when the electron reaches the end of the tunnel $2mi = 3218.6m$ the velocity should allow for (☆) $E = \gamma m = \sqrt{m^2+\vec{p}^2} = 40\,GeV$, $\left[|\vec{p}| = Ft = e|\vec{E}|t\right]$

from $X(t)$ we get

$t = \frac{m}{F}\sqrt{\left(\frac{F}{m}X+1\right)^2-1} = \frac{m}{e|\vec{E}|}\sqrt{\left(\frac{e|\vec{E}|}{m}(3218m)+1\right)^2-1}$, Putting back into ☆

$(40\,GeV)^2 = m^2 + e^2|\vec{E}|^2t^2 = m^2 + \cancel{e^2|\vec{E}|^2}\frac{m^2}{\cancel{e^2|\vec{E}|^2}}\left[\left(\frac{e|\vec{E}|}{m}(3218m)+1\right)^2-1\right]$

Solving for $|\vec{E}| = \left[\sqrt{\frac{E^2-m^2}{m^2}+1}-1\right]\frac{m}{e(3218)} = \left(\frac{E}{m}-1\right)\frac{m}{e(3218)}$

$= \frac{(40\cdot10^3\,MeV - \frac{1}{2}MeV)}{(3218\,m)(1.6\times10^{-19}C)} = \frac{4\cdot10^4\,\cancel{MeV}}{5\times10^{-6}\,C\,m}\left(\frac{1\,kg\frac{m^2}{s^2}}{6.24\cdot10^{12}\,MeV}\right) = \boxed{1.24\cdot10^7\,\frac{N}{C}}$