Hartle solutions week 5

5-13. *[B,S] One reaction for photoproducing pions is*

$$\gamma + p \to n + \pi^+ \, .$$

Find the minimum energy (the threshold energy) a photon would have to have to produce a pion in this way in the frame in which the proton is at rest. Is this energy within reach of contemporary accelerators?

Solution: The threshold condition, Ref. (b) in Box 5.1 just needs to be evaluated in the frame in which the proton is at rest. In that frame

$$p_p^\alpha = (m_p, 0, 0, 0)$$

and the condition gives

$$-2E_\gamma m_p - m_p^2 = -(m_n + m_\pi)^2$$

for the threshold energy E_γ of the photon. Solving for E_γ gives the approximation for $m_n \approx m_p$

$$E_\gamma = m_\pi \left(1 + \frac{m_\pi}{2m_p}\right) \approx 150\text{MeV}.$$

5-15. *[C] A source and detector are spaced a certain angle ϕ apart on the edge of a rotating disk. The source emits radiation at a frequency ω_* in its instantaneous rest frame. What frequency is the radiation detected at? Hint: Little information is given in this problem because little is needed.*

Solution: From the symmetry of the problem the angle the detected radiation makes with the emitter's velocity is the π minus the angle it makes with the detectors velocity. The velocity of the emitter and detector are the same. Viewed from the inertial frame in which the center of the disk is at rest, the red shift on emission is canceled by the blue shift on detection from (5.73). The detected frequency is ω_*.

5-22. *[C] (The Relativistic Rocket) A rocket accelerates by ejecting part of its rest mass as exhaust. The speed of the exhaust is a constant value u in the rocket's rest frame. Use the conservation of energy and momentum to find the ratio of final to initial rest mass for a rocket that accelerates from rest to a speed V. [Hint: Rest mass is not conserved — energy and momentum are conserved. You might want to start by working the same problem in Newtonian mechanics.]*

Solution: *(The Relativistic Rocket)* Let M be the rest mass of the rocket and V its velocity. It's easiest to first work out the conservation of energy and momentum in the rest frame of the rocket and then transform back to a frame where it's moved with speed V. In the rest frame, suppose ΔM of the rest mass (counted negative) is ejected at speed u. Let $\Delta V'$ be the resulting change in speed of the rocket and ΔM_e the rest mass of the ejecta. To first order in small quantities conservation of energy and momentum are:

$$\begin{aligned} M &= M + \Delta M + \Delta M_e \gamma_u \\ 0 &= M\Delta V' - \Delta M_e u \gamma_u \end{aligned}$$

Eliminating ΔM_e we get

$$\Delta V' = -(\Delta M/M)\, u\,.$$

To transform back to the charge ΔV in the frame where the rocket is moving with speed V, we use the addition of velocity formula (3.40a) to find

$$V + \Delta V = \frac{\Delta V' + V}{1 + V\Delta V'} \approx V + \Delta V'\left(1 - V^2\right)\,.$$

Then

$$\frac{\Delta V}{1-V^2} = -u\frac{\Delta M}{M} \, .$$

Integrating both sides gives

$$\tanh^{-1} V = +u \log\left(\frac{M_0}{M}\right)$$

where M_0 is the initial rest mass. With a little algebra one gets

$$\frac{M}{M_0} = \left(\frac{1-V}{1+V}\right)^{\frac{1}{2u}} \, .$$

As the rest mass is used up, V approaches the velocity of light.

5-23. *[C] (Tachyons)*

a) Tachyons would travel on *spacelike* curves and the distinction between spacelike, timelike, and null is Lorentz invariant.

b) Let $x^\alpha(\lambda)$ be the world line of a tachyon, and $u^\alpha = dx^\alpha/d\lambda = (dx^\alpha/dt)(dt/d\lambda)$. Then

$$\mathbf{u}\cdot\mathbf{u} = \left(\frac{dt}{d\lambda}\right)^2 \left[-1+\vec{V}^2\right] \, .$$

This is greater than zero if $|\vec{V}| > 1$. Thus, s is a good parameter and $\mathbf{u}\cdot\mathbf{u} = 1$.

c)

$$\begin{aligned} u^t &\equiv \frac{dt}{ds} = \frac{1}{\sqrt{\vec{V}^2 - 1}} \\ \vec{u} &= \frac{d\vec{x}}{dt}\frac{dt}{ds} = \frac{\vec{V}}{\sqrt{\vec{V}^2-1}} \end{aligned}$$

d) If $\mathbf{p} \equiv m\mathbf{u}$, then $\mathbf{p}^2 = m^2 = -E^2 + \vec{p}^{\,2}$, so that

$$E = \pm\sqrt{\vec{p}^{\,2} - m^2} \, .$$

e) Since the four-momentum is spacelike, there is always some frame in which its t component is negative.

$$E' = \gamma(E - vp)$$

But if $E > 0$, then $|E| < |p|$ from (d) so for a sufficiently large v of the right sign $E' < 0$.

f) Consider a decay of a particle A into a tachyon T and another particle B.

$$\mathbf{p}_A = \mathbf{p}_B + \mathbf{p}_T \, .$$

Examine the decay in the rest frame of the particle A initially. Then from conservation of momentum

$$\vec{p}_B = -\vec{p}_T \equiv \vec{p} \, .$$

Conservation of energy would mean

$$m_A = \sqrt{p^2 + m_B^2} - \sqrt{p^2 - m_T^2}$$

where we have assumed the tachyon has negative energy. Energy is conserved if this equation can be satisfied for some p. At $p = m_T$, the smallest possible value, the right hand side is $\sqrt{m_T^2 + m_B^2}$ which is greater than m_A. At $p = \infty$ the right hand side is 0. So somewhere in between there will be a value for which it is equal to m_A.