Hartle solutions week 6

2-3. *[C] (a) Verify the relation (2.4) between the sum of the interior angles of a spherical triangle and its area when two of the angles are right angles.*

(b) Prove the relation generally.

Solution:

a) Such a triangle can be bounded by the equator and two lines of longitude differing by an angle α. The area A is $(\alpha/2\pi) \times$ (area of a hemisphere) $=$ αa^2.

$$\begin{pmatrix}\text{sum of the}\\ \text{interior angles}\end{pmatrix} = \pi + \alpha = \pi + \frac{(\alpha a^2)}{a^2} = \pi + \frac{A}{a^2}\,.$$

b) The three great circles that bound a spherical triangle divide the sphere up into eight triangles. Any two circles divide the sphere into wedges whose opening angle is one of the interior angles of a triangle, and whose area is the sum of the areas of two of the triangles. This gives a set of relations of the form

$$A + A' = \frac{1}{2\pi}\begin{pmatrix}\text{interior}\\ \text{angle}\end{pmatrix} \cdot 4\pi a^2$$

which could be solved for the areas of the triangles in terms of their interior angles.

However, it is not necessary to carry out this solution. Arguments of symmetry and some special cases are enough to find the result. The above relations show that the area of a spherical triangle are *linearly* related to the three interior angles: α, β, γ. Since, in a general triangle, no one of these angles is preferred over any other, the area must be related linearly and symmetrically to the angles by a relation of the form

$$A = c(\alpha + \beta + \gamma) + d$$

with constants c and d depending on a to be determined. The special case considered in (a) with $\beta = \gamma = \pi/2$ gives

$$A = c(\pi + \alpha) + d = \alpha a^2$$

holding for arbitrary α. Thus, $c = a^2$ and $d = -a^2\pi$, giving

$$A = a^2(\alpha + \beta + \gamma - \pi)$$

or, what is the same thing:

$$\alpha + \beta + \gamma = \pi + \frac{A}{a^2}.$$

2-5. *Calculate the area of a circle of radius r (distance from center to circumference) in the two - dimensional geometry which is the surface of a sphere of radius a. Show that this reduces to* πr^2 *when* $r \ll a$.

Solution: Refer to Fig.2.6 for the geometry. Consider an element of area at (θ, ϕ) spanned by coordinate intervals $(d\theta, d\phi)$. The length of the edge of size $d\theta$ is $ad\theta$, the length of the edge of size $d\phi$ in $a \sin\theta d\phi$. Since the coordinate lines are orthogonal the area is

$$(ad\theta)(a \sin\theta d\phi) .$$

The circle of radius r lies at $\theta = r/a$. Integrating the element of area above

$$A = \int_0^{r/a} d\theta \int_0^{2\pi} d\phi \; a^2 \sin\theta d\theta d\phi$$

gives the result:

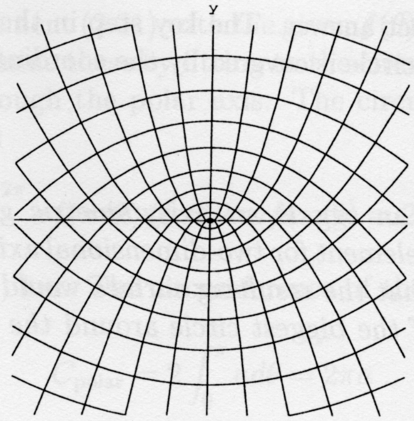
$$A = 2\pi a^2[1 - \cos(r/a)] .$$

For small r/a

$$\cos\left(\frac{r}{a}\right) \;=\; 1 - \frac{1}{2}\left(\frac{r}{a}\right)^2 + O\left(\frac{r}{a}\right)^4$$

$$A = \pi r^2 \;+\; O\left(\frac{r}{a}\right)^4.$$

Problem2.7



b)

$$\begin{aligned} dS^2 &= dx^2 + dy^2 \\ &= (\mu d\nu + \nu d\mu)^2 + (\mu d\mu - \nu d\nu)^2 \\ dS^2 &= \left(\mu^2 + \nu^2\right)\left(d\mu^2 + d\nu^2\right) \end{aligned}$$

c) The curves intersect at right angles because there are no cross terms $d\mu d\nu$ in the metric.

d) The equation of a circle is $x^2 + y^2 = r^2$ which becomes

$$\begin{aligned} \mu^2\nu^2 + \frac{1}{4}\left(\mu^2 - \nu^2\right)^2 &= r^2 \\ \frac{1}{4}\left(\mu^2 + \nu^2\right)^2 &= r^2 \\ \mu^2 + \nu^2 &= 2r \end{aligned}$$

e) The circumference C is

$$\begin{aligned} C = \oint dS &= \oint \left(\mu^2 + \nu^2\right)^{\frac{1}{2}} \left(d\mu^2 + d\nu^2\right)^{\frac{1}{2}} \\ C &= (2r)^{\frac{1}{2}} \oint d\mu \left(1 + \left(\frac{d\nu}{d\mu}\right)^2\right)^{\frac{1}{2}} \\ &= (2r)^{\frac{1}{2}} \int_{-\sqrt{2r}}^{+\sqrt{2r}} d\mu \left[1 + \frac{\mu^2}{(2r - \mu^2)}\right]^{\frac{1}{2}} \\ &= 2\pi r \end{aligned}$$

This is, of course, the correct answer. The key step in the above evaluation is recognizing that the whole circle is covered by the coordinate range $\mu = -\sqrt{2r}$ to $\mu = \sqrt{2r}$.

Problem2.19

Solution: The line element (2.21) with the given $f(\theta)$ depends on two parameters a and ϵ. We can determine these by fitting to the circumferences of the equator and a great circle through the polar axis. The circumference of the equator $\theta = \pi/2$ from (2.21) is

$$C_{\rm eq} = \int_0^{2\pi} af(\pi/2)d\phi = 2\pi af(\pi/2) = 2\pi(1+\epsilon)a\ .$$

This must be $2\pi(6378)$ km. The circumference of the great circle $\phi = 0$ from (2.21) is

$$C_{\rm polar} = 2\int_0^{\pi} ad\theta = 2\pi a\ .$$

This must be $2\pi(6357)$ km. Thus

$$a = 6357\ \text{km}, \quad \epsilon = .003$$

Problem3.4

Solution: The mass density in the sphere is

$$\mu = \frac{M}{(4\pi R^3/3)} = \frac{3M}{4\pi R^3} = \text{const.} \tag{1}$$

The spherical symmetry of the problem implies that Φ is a function only of the radius r. Poisson's equation (3.18) for the gravitational potential is

$$\frac{1}{r^2}\frac{1}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\mu. \tag{2}$$

The general solution inside ($r < R$) which does not diverge at $r = 0$ is

$$\Phi(r) = Ar^2 + B \tag{3}$$

where A and B are constants. A is determined from (2) and (1) to be

$$A = \frac{2\pi}{3}\ G\mu = \frac{1}{2}\left(\frac{GM}{R}\right)\frac{1}{R^2}. \tag{4}$$

B is determined by matching to the exterior solution $\Phi(r) = -GM/r$ at $r = R$ to be $B = -3GM/(2R)$. Thus,

$$\Phi(r) = \frac{GM}{2R}\left[\left(\frac{r}{R}\right)^2 - 3\right], \quad r < R \tag{5}$$

$$= -\frac{GM}{r}, \quad r > R. \tag{6}$$