Hartle week 6 solutions part 2

Solution: Lagrange's equations

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)+\frac{\partial L}{\partial x}=0 \tag{1}$$

are the necessary condition for an extremum to functionals of the form

$$S\left[x(t)\right]=\int dt\, L\left(\dot{x},x\right)\ . \tag{2}$$

In the present case, $L=\dot{x}^2+x^2$ and the Lagrange equation (1) is

$$\ddot{x}_{cl}=x_{cl}$$

whose general solution is a linear combination of $\sinh t$ and $\cosh t$. The solution satisfying $x(0)=0$, $x(T)=1$ is

$$x_{cl}(t)=\frac{\sinh t}{\sinh T}\ .$$

The value of the action at this extremum can be found by doing the integral directly, but is most easily computed by integrating (2) by parts to give

$$S[x(t)]=\dot{x}(t)\,x(t)\Big|_0^T+\int_0^T dt\,x(t)\ \left[-\ddot{x}(t)+x(t)\right]\ .$$

The second term vanishes because of Lagrange's equation, so

$$S\left[x_{cl}(t)\right]=\coth T \tag{3}$$

for the extremal path. The argument of the action is positive for any choice of $x(t)$ and can be made arbitrarily big by choosing a wiggly path with big $\dot{x}$. The extremum, therefore, cannot be a maximum but must be a minimum. For example, the simple path

$$x_*(t)=\frac{t}{T}$$

satisfies the boundary conditions, and

$$S\left[x_*(t)\right]=\frac{1}{T}\left(1+\frac{T^2}{3}\right)$$

which is greater than (3), as calculating a few values or a simple plot will show.

3-6. *[B,E,C] Estimate the gravitational self-energy of the Moon as a fraction of the Moon's rest mass energy. Is this ratio larger or smaller than the accuracy of the Lunar laser ranging test of the equality of gravitational and inertial mass?*

Solution: The gravitational self energy is of order

$$E_{\text{self}} \sim \frac{GM_{\text{moon}}^2}{R_{\text{moon}}}$$

and the ratio to the rest energy is

$$\frac{E_{\text{self}}}{E_{\text{rest}}} \sim \frac{GM_{\text{moon}}}{R_{\text{moon}}c^2} \sim \frac{(6.67\times10^{-8})(7.35\times10^{25})}{(1.7\times10^{8})(3\times10^{10})^2} \sim 3\times10^{-11}$$

which is within the 10^{-13} accuracy of lunar laser ranging.

6-3. *[S] Assuming that the acceleration is the acceleration of gravity at the surface of the Earth, how wide does the elevator in Figure 6.5 have to be for the light ray to fall by 1 mm over the course of its transit? Is this a thought experiment that could be realized on the surface of the Earth?*

Solution: To fall 1 mm with the acceleration of gravity at the surface of the Earth, $g = 980\ \text{cm/s}^2$, requires a time $t_f = \sqrt{2(1\ \text{mm})/g} \approx .01$ s. During this time the light ray will travel a horizontal distance of approximately (.01 s) $c \approx$ 3000 km. The laboratory needs to be this wide. Thus it is a significant fraction of the radius of the Earth so the experiment couldn't be carried out there.

proble

Solution: To make the algebra more manageable, choose $m = 1$ for the mass of the ball, put $G = 1$, and let M be the mass of the Earth. Orbits in Newtonian mechanics are characterized by an energy $\mathcal{E}$ and an angular momentum ℓ. The ball in question starts a distance s further out than the radius of a circular orbit R, but moving with the same velocity

$$V = \left(\frac{M}{R}\right)^{\frac{1}{2}}$$

as a particle in that circular orbit.

The energy $\mathcal{E}$ and angular momentum ℓ are therefore

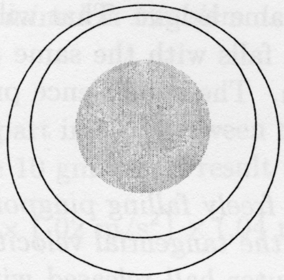
$$\mathcal{E} = \frac{1}{2}V^2 - \frac{M}{(R+s)} \quad , \quad \ell = V(R+s). \tag{1}$$

m6.5

The eccentricity ϵ of an orbit with these parameters is given by

$$\epsilon = \left[1 + \frac{2\mathcal{E}\ell^2}{M^2}\right]^{\frac{1}{2}} = \frac{s}{R}. \tag{2}$$

The figure below shows the two orbits for $\epsilon = 1$. The distance between the orbits changes significantly over one transversal. That supports the conclusion in Example 6.3 that there is significant change in the distance between the particles over one period.



6-8. *[S] It's not legitimate to mix relativistic with non-relativistic concepts, but imagine that that a photon with frequency ω_* is like a particle with gravitational mass $\hbar\omega_*/c^2$ and kinetic energy $K = \hbar\omega$. Using Newtonian ideas, calculate the "kinetic" energy loss to a photon that is emitted from the surface of a spherical star of radius R and mass M and escapes to infinity. From this calculate the frequency of the photon at infinity. How does this compare with the gravitational redshift in (6.14) to first order in $1/c^2$?*

Solution: In Newtonian physics, the loss in kinetic energy of a particle that moves from the surface of the star to infinity is equal to the difference in potential energy between those locations. That difference is GmM/R for a particle of mass m. Thus, if K_R is the kinetic energy at R and K_∞ the kinetic energy at infinity,

$$K_R - K_\infty = GmM/R\,.$$

Equating $K = \hbar\omega$ and $m = \hbar\omega/c^2$ for a photon, we find

$$\omega_\infty = \omega_R\left(1 - \frac{GM}{c^2R}\right)\,.$$

This is exactly the gravitational redshift to first order in $1/c^2$.