1) $\vec{V} = \vec{r} = \langle x, y, z \rangle \qquad \tau = x^2+y^2+z^2 \le 1$

WTS: Div Thm holds true

a) $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \nabla\cdot\langle x,y,z\rangle\, r^2 \sin\theta \, dr\, d\theta\, d\phi$

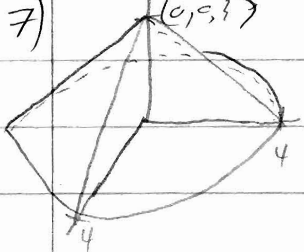
$3\cdot \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2\sin\theta\, dr\, d\theta\, d\phi = 3\cdot\frac{4}{3}\pi = 4\pi$

b) $\vec{n} = \hat{r} = \langle 0,0,1\rangle$

$\vec{r} = \langle \cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta\rangle$

$\vec{r}\cdot\hat{n} = \sin\theta$

$\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = 2\pi\int_0^{\pi} \sin\theta\, d\theta = 2\pi\left[-\cos\theta\right]_0^{\pi} = 2\pi(1+1) = 4\pi$

7) 

$\Rightarrow 0\le r\le 4-\frac{4z}{3} \qquad 0\le z\le 3 \qquad 0\le\phi\le 2\pi$

$\vec{V} = \langle x,y,z\rangle$

$\nabla\cdot\vec{V} = 3$

$\int_0^{2\pi}\int_0^{3}\int_0^{4-\frac{4z}{3}} 3r\,dr\,dz\,d\phi$

$6\pi\int_0^{3}\int_0^{4-\frac{4z}{3}} r\,dr\,dz$

$3\pi\int_0^{3} \left(16 - 32\frac{z}{3} + \frac{16z^2}{9}\right) dz$

$3\pi\left[16z - \frac{16z^2}{3} + \frac{16z^3}{27}\right]_0^3$

$3\pi\left[48-48+16\right] = 48\pi$

10) $\vec{V} = \langle y, xz, 2z-1 \rangle$

find $\iint_\sigma \vec{V}\cdot\vec{n}\, d\sigma$ over the curved surface of the hemisphere $x^2+y^2+z^2=9$.

this is equal to $\int_\tau \nabla\cdot\vec{V}\, d\tau - \int_\sigma \vec{V}\cdot\langle 0,0,-1\rangle\, d\sigma$

$\nabla\cdot\vec{V} = 2$ $\qquad$ $\vec{V}\cdot\vec{n} = -2z+1$ $\quad$ w/ $z=0$

$2\int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} r^2 \sin\theta\, dr\, d\theta\, d\phi - 1\cdot\int_0^{2\pi}\int_0^{3} r\, dr\, d\theta$

$2\cdot\frac{1}{2}\cdot\frac{4}{3}\cdot\pi\cdot 27 - 1\cdot 9\pi = 9\pi(4-1) = 27\pi$

16) a) WTS: $\int_\tau (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\, d\tau = \oint_\sigma (\phi\nabla\psi)\cdot n\, d\sigma$

Since ϕ is a scalar funct & $\nabla\psi$ is a vector funct, $\phi\nabla\psi$ is a vector funct. let $\vec{V} = \phi\nabla\psi$. Then $\nabla\cdot\vec{V} = \nabla\cdot(\phi\nabla\psi) = \phi\nabla\cdot\nabla\psi + \nabla\psi\cdot\nabla\phi =$ $\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$. Apply the Div. Thm we get

$$\oint_\sigma (\phi\nabla\psi)\cdot\vec{n}\, d\sigma = \int_\tau \nabla\cdot(\phi\nabla\psi)\, d\tau = \int_\tau (\phi\nabla^2\psi + \nabla\phi\nabla\psi)\, d\tau$$

as desired

QEMFD

b) WTS: $\oint_\sigma (\phi\nabla\psi - \psi\nabla\phi)\, d\sigma = \int_\tau (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d\tau$

Apply part (a) to the surface integral and recalling that integration is linear opperator we get

$$\int_\tau (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\, d\tau - \int_\tau (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)\, d\tau = \oint_\sigma (\phi\nabla\psi - \psi\nabla\phi)\, d\sigma$$

$$\int_\tau (\phi\nabla^2\psi + \nabla\phi\nabla\psi - \psi\nabla^2\phi - \nabla\psi\cdot\nabla\phi)\, d\tau =$$

$$\int_\tau (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d\tau =$$

As desired since the dot product of two vectors is commutative.

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