Non-Linear Dynamics Homework Solutions

Week 7

February 23, 2009

7.3.1 Consider the system

$$\begin{aligned}\dot{x} &= x - y - x(x^2 + 5y^2)\\ \dot{y} &= x + y - y(x^2 + y^2).\end{aligned}$$

a) Classify the fixed point at the origin.
We first compute the Linearization matrix using the Jacobian

$$J|_O = \begin{pmatrix} 1 - 3x^2 - 5y^2 & -1 - 10yx \\ 1 - 2yx & 1 - x^2 - 3y^2 \end{pmatrix}_O = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

which has $T = 2$ and $D = 2$, implying that the origin is an unstable spiral since $T^2 - 4D = -4 < 0$.

b) Rewrite the system in polar coordinates using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.
From the first equation we get

$$\begin{aligned}r\dot{r} &= x^2 - xy - x^2(x^2 + 5y^2) + yx + y^2 - y^2(x^2 + y^2)\\ &= x^2 + y^2 - x^4 - y^4 - 6x^2y^2\\ &= r^2 - ((x^2 + y^2)^2 - 2x^2y^2) - 6x^2y^2\\ &= r^2 - r^4 - 4r^4\cos^2\theta\sin^2\theta\end{aligned}$$

Note that in this simplification we have used the facts that $x^2 + y^2 = r^2$, $x = r\cos\theta$ and $y = r\sin\theta$. Dividing this final equation through by r^2we find that

$$\dot{r} = r - r^3(1 + 4\cos^2\theta\sin^2\theta).$$

We also find in a similar manner that

$$\begin{aligned}r^2\dot{\theta} &= (x^2 + yx - yx(x^2 + y^2)) - (yx - y^2 - yx(x^2 + 5y^2))\\ &= r^2 + 4xy^3\\ \dot{\theta} &= 1 + 4r^2\cos\theta\sin^3\theta\end{aligned}$$

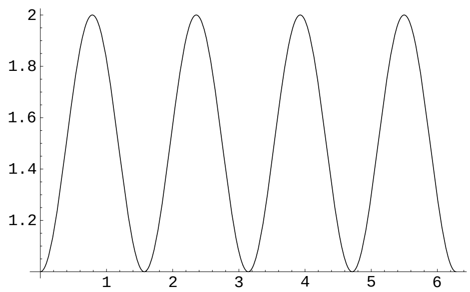

Figure 1: Plot of the periodic term from 7.3.1 vs. θ

c) Determine the circle of maximum radius, r_1, centered on the origin such that all trajectories have a radially *outward* component on it.

This task is equivalent to finding the maximum radius for which $\dot{r} > 0$ for all θ. If we look at our equation for $\dot{r}$, we can see that the bigger the periodic term $(1+4\cos^2\theta\sin^2\theta)$ is, the harder it will be to make $\dot{r} > 0$, so we want to find the maximum of the term inside the parenthesis, since we'll our r_1 to make $\dot{r} > 0$ for that maximum, but will also be guarenteed that $\dot{r} > 0$ for all other θ values, since they would only make the periodic term less, and hence, make $\dot{r}$ even larger. To find this maximal value, we could resort to analytic techniques, ie. find the derivitive of the periodic term with respect to θ, set that equal to zero and then figure out which of the resulting critical points give us maxima and which give us minima. There may even be an easier analytic method involving geometric of trigonometric aproaches, but a graphic argument is really easy and quite convincing. Taking a look at Figure 4, we can se that the maximum value of this term is 2. We can therefore set that term equal to 2 in our equation and see what values of r makes $\dot{r}$ positive. This brings us to solving the inequality

$$\begin{aligned} 0 < \dot{r} &= r - 2r^3 \\ &= r(1 - 2r^2) \end{aligned}$$

Since this expression for $\dot{r}$ is a downward facing cubic with x intercepts at $r = \pm\sqrt{1/2}$ and $r = 0$ we can deduce that for all $r < \sqrt{1/2}$ but greater than zero, $\dot{r} > 0$. See Figure 5 for graphical clearification on this arguement. Thus we get the desired value by setting $r_1 = \sqrt{1/2} - \epsilon$ for any positive but small as we like ϵ.

d) Determine the circle of minimum radius, r_2, centered on the origin such that all trajectories have a radially *inward* component on it.

We proceed here as we did in part (c), only here we want the smallest r for which $\dot{r}$ is negative for all θ. Consequently, we want to find the minimum of $(1 + 4\cos^2\theta\sin^2\theta)$ instead of the maximum. By applying analytic methods such as those described in part (c), or looking at Figure 4 again, we can see that the minimum of the term is 1. We use this in the equation for $\dot{r}$ as we did above, and find that if $\dot{r} = r(1-r^2)$ then we will have a negative value of $\dot{r}$ so long as $r > 1$. Thus we may set $r_2 = 1 + \epsilon$ for some small but positive ϵ.

e) Prove that the system has a limit cycle somewhere in the trapping region $r_1 < r < r_2$.

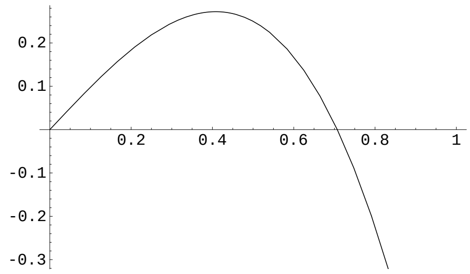

Figure 2: Plot of $\dot{r}$ versus r

Proof. By our construction we have a trapping region that satisfies the conditions of the Poincaré-Bendixson Theorem. Consequently, there exists a limit cycle in the region in question. $\square$

7.3.7 Consider the system

$$\begin{aligned} \dot{x} &= y + ax(1 - 2b - r^2) \\ \dot{y} &= -x + ay(1 - r^2) \end{aligned}$$

where a and b are parameters ($0 < a \leq 1$, $0 \leq b < 1/2$) and $r^2 = x^2 + y^2$.

a) Rewrite this system in polar coordinates.
We proceed as we did in problem 7.3.1.

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= (ax^2 + ay^2)(1 - r) - 2bax^2 \\ \dot{r} &= ar(1 - r^2 - 2b\cos^2\theta) \end{aligned}$$

$$\begin{aligned} r^2\dot{\theta} &= -x^2 + ayx(1 - r^2) - y^2 - ayx(1 - 2b - r^2) \\ \dot{\theta} &= -1 + 2ba\cos\theta\sin\theta \end{aligned}$$

b) Prove that there is at least one limit cycle, and that if there are several, they all have the same period $T(a, b)$.
As in problem 7.3.1, we find r_1 and r_2 with the properties we need to have a trapping region. We want to maximaize the periodic term $2b\cos^2\theta$, which happens when $\theta = k\pi$ for some integer k. At these points the term in question equals $2b$, so we substitute this into our equation for $\dot{r}$ and find that $\dot{r} = (1 - r^2 - 2b) = ar((1 - 2b) - r^2)$, so if we set $r_1 = \sqrt{1 - 2b} - \epsilon$, for some small but positive ϵ we get what we need.
Since the minimum of the periodic term is 0, we can use this to find a suitable r_2. We get $\dot{r} = ar(1 - r^2)$ so if we let $r_2 = 1 + \epsilon$ for some small but positive ϵ, then we have our outer trapping region boundary.

By the Poincaré-Bendixson Theorem, and the existence of a trapping region for this system, the system must have at least one limit cycle in the region inbetween r_1 and r_2. Reguarding the period of such trajectories, note that $\dot{\theta}$ depends only on θ and not on r. Consequently, for any two initial conditions with the same initial θ value, the trajectories will make a full revolution in the same amount of time, since their angular positions are governed by the same dynamical equations.

c) Prove that for $b = 0$ there is only one limit cycle.

Proof. When $b = 0$ our inner trapping radius will be $r_1 = 1 - \epsilon$ and our outer will be $r_2 = 1 + \epsilon$. For two limit cycles to exist, there must be a radial distance d_r between them for any given value of θ, but we can make the values of ϵ as low as we like, so low that $2\epsilon < d_r$ so that there is no way both could fall inside the trapping region. □

7.5.1 For the van der Pol oscillator with $\mu >> 1$, show that the positive branch of the cubic nullcline begins at $x_A = 2$ and $x_B = 1$.

The start of the positive branch begins at the local minimum on the positive side of the graph, and is considered ended at the point along that branch which is the same height as the local maximum on the negative branch (See Figure 7.5.1 in Strogatz). So, to find x_A, we take the derivative of the cubic nullcline and find where it is equal to zero, since this locates the local minimum of the nullcline. Since $F'(x) = x^2 - 1$, our critical points are at ± 1, so $x_A = 1$. Now we figure out how high the local maximum of the cubic nullcline is: $F(-1) = -1/3 + 1 = 2/3$. Now we set $F(x) = 2/3$ and find the other solution (the one that is not $x = -1$). This can be done by plugging the equality into mathematica, or by knowing that if we divide the polynomial $F(x) - 2/3 = 0$ by $(x+1)$ then it leaves us with a quadratic polynomial which will have all of the roots other than $x = -1$. Choosing the former method, we find that $x = 2$ is also a solution to $F(x) - 2/3 = 0$, and so $x_B = 2$.

7.5.3 Estimate the period of the limit cycle of $\ddot{x} + k(x^2 - 4)\dot{x} + x = 0$ for $k >> 1$.

We shall solve this by defining $z = x - 1$, which gives us a form in some ways more similar to what is given in the book, but in others is a bit different. Doing this, we find that

$$\ddot{z} + k((z+1)^2 - 4)\dot{z} + z = 0.$$

Then following the process of example 7.5.1, we notice that since

$$\ddot{z} + k\dot{z}((z+1)^2 - 4) = d/dt(\dot{z} + k[\frac{1}{3}(z+1)^3 - 4z]),$$

we can set

$$F(z) = \frac{1}{3}(z+1)^3 - 4z$$

so that if we let $w = \dot{z} + kF(z)$, then

$$\dot{w} = \ddot{z} + k\dot{z}((z+1)^2 - 4) = -z.$$

(Note that the difference between the way we did it here and the way we could have done had we left things in terms of x is that if we had left things, we would have $\dot{w} = 1 - z$ instead). Now we let $y = w/k$ so that $\dot{z} = k[y - F(z)]$ and $\dot{y} = -z/k$.

Now, the period is given by

$$T = 2\int_{t_A}^{t_B} dt,$$

since the time of the period is approximately twice the time it takes to get through just one of the slow branches. Now since

$$\frac{dy}{dt} \approx F'(z)\frac{dz}{dt} = ((z+1)^2 - 4)\frac{dz}{dt}.$$

Now since $dy/dt = -z/k$ we get $dz/dt = -z/(k((z+1)^2 - 4)$, from which it follows that

$$dt \approx -\frac{k((z+1)^2-4)}{z}dz.$$

Now we must find the limits of integration. We do this just as we did for the last excercise. Since $F'(z) = z^2 + 2z - 3$ has zeroes at $z = 1$ and $z = -3$. We want the positive one of these for our z_A value, and for z_B we see that $F(-3) = 9 + 1/3$. The value of z_B will be the other value of z for which $F(z) = 9 + 1/3$. Solving this equation (using mathematica or polynomial division) for z we find that if $z \neq -3$ then $z = 3$. Thus $z_B = 3$ so we must integrate from 3 to 1 (because of the direction of the flow along this part of the nullcline we want the upper limit to be the smaller one). Thus we find using mathematica to integrate for us that

$$\begin{aligned} T &\approx -2k\int_3^1 \frac{(z+1)^2-4}{z}dz \\ &= 2k(8 - \ln 27). \end{aligned}$$

7.5.6 (Biased van der Pol) Suppose the van der Pol oscillator is biased by a constant force: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$, where a is some real parameter and $\mu > 0$ as usual.

a) Find and classify all the fixed points.
We first write out our system in Liénard form. First we define $F(x)$ so that

$$\ddot{x} + \mu(x^2-1)\dot{x} = \frac{d}{dx}(\dot{x} + \mu F(x)),$$

which works out if $F(x) = x^3/3 - x$, just as it did in the non-biased case. Then everything works out the same as it did in the non biased case (see Example 7.5.1 from Strogatz), only that in our case, working everything through as is the afore mentioned example, we find that $\dot{y} = -(x-a)/\mu$. Thus, our system becomes

$$\begin{aligned} \dot{x} &= \mu[y - F(x)] \\ \dot{y} &= -\frac{x-a}{\mu} \end{aligned}$$

Consequently, our x-nullcline becomes $y = F(x)$ and our y-nullcline becomes $x = a$. Thus our intersection and only fixed point is at $x^* = (a, F(a)) = (a, a^3/3 - a)$. Next we compute the Jacobian matrix and evaluate at $x*$

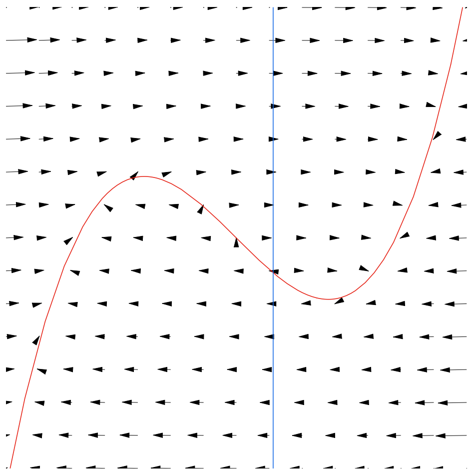

Figure 3: Plot of the nullclines for problem 7.5.6

$$J|_{x^*} = \begin{pmatrix} -\mu(x^2-1) & \mu \\ -1/\mu & 0 \end{pmatrix}_{x^*} = \begin{pmatrix} -\mu(a^2-1) & \mu \\ -1/\mu & 0 \end{pmatrix}.$$

From this we find that our trace and determinent are going to be given by $T = -\mu(a^2-1)$ and $D = 1$. Since our determinent is always positive we are never going to have any sort of saddlish behaviour. Furthermore, we have that the system is unstable when $-1 < a < 1$ and unstable otherwise. Furthermore, for sufficiently high values of a the value of $T^2 - 4D = \mu^2(a^2-1)^2 - 4$ will be positive, implying that the fixed point will be a node. Clearly, this can't always be the case, since plugging in $a = 1$ readily checks out as giving us a negative value for $T^2 - 4D$, impplying that we do have spiral behaviour for certain values of a.

b) Plot the nullclines in the Liénard plane. Show that if they intersect on the middle branch of the cubic nullcline, the corresponding fixed point is unstable.

See Figure 1 for a numerically produced plot given that $a = .4$ and $\mu = 5$, which has been laid on top of a vector field of that system.

To see that the fixed point in cases like these are unstable, we resort to the work we did for part (a) with respect to finding stabilities of fixed points for various regions.

c) For $\mu >> 1$, show that the system has a stable limit cycle if and only if $|a| < a_c$, where a_c is to be determined. (Hint: Use the Liénard plane.)

For $|a| < a_c = 1$, the y-nullcline intersects the x-nullcline on the middle branch of the cubic nullcline. We know that when this is the case the intersection (fixed point), is unstable, so all initial conditions move away from from that point. Furthermore, they all move toward the cubic nullcline, and once there, they will move up along the nullcline if it has reached the negative leg, and will move down if on the positive leg. Eventually, the trajectory will reach either a local maximum or a local minimum (depending on which leg you are on). I for instance, you are on the positive leg and get to the minimum, you will move just past where the cubic nullcline is keeping you from quickly shooting past it and into the negative x realm, but eventually it will hit the negative leg of the cubic

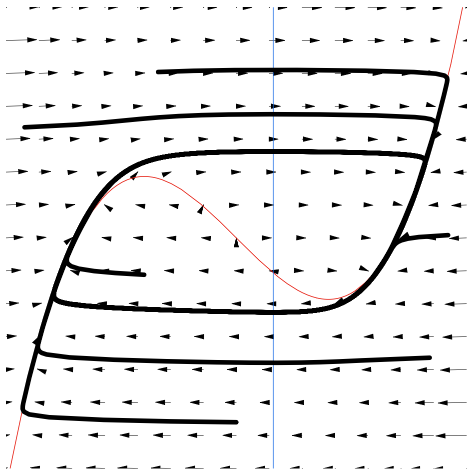

Figure 4: Plot of the nullclines for problem 7.5.6

nullcline, and then move slowly upwards, then hit the peak and move quickly over to the positive leg, and then repeat. See Figure 2 for a numerical example.

If, however $|a| > a_c$, there can be no stable limit cycle, since the x nullcline intersects the cubic nullcline along one of the positive branches, and consequently, once a trajectory hits that nullcline, it will move slowly along it and toward the intersection (since if it hits below the intersection, then $x < a$ so $\dot{x}$ is positive and similarly negative if $x > a$). Thus since all trajectories end up moving slowly along nullclines and toward the fixed point, there can be no stable limit cycles. See Figures 3 and 4 for examples.

d) Sketch the phase portrait for a slightly greater than a_c. Show that the system is *exciteable*.
See Figure 5 for the phase portrait in the case when $a = 1.1$. Notice that all of the trajectories end up getting stuck near the local minimum of the cubic nullcline. Now, if a slight distubance moves the state of the system to just below the fixed point then it will shoot accross to the negative leg of the cubic, and then go along the whole of the cycle but get stuck again at the globally attracting rest state.

7.6.2 (Calibrating regular perturbation theory) Consider the initial value problem $\ddot{x} + x + \epsilon x = 0$, with $x(0) = 1$, $\dot{x}(0) = 0$.

a) Obtain the exact solution to the problem.
We rearange to get the equation $\ddot{x} = -(1+\epsilon)x$, which is easily seen, given the initial conditions, to have the solution $x(t) = \cos(\sqrt{1+\epsilon}t)$.

b) Using regular perturbation theory, find x_0, x_1, and x_2 in the series expansion $x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + O(\epsilon^3)$.
We find that for the differential equation to hold for the expansion, we must have that

$$\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 = -(1+\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2).$$

From this it follows that

$$[\ddot{x}_0 + x_0] + [\ddot{x}_1 + x_0 + x_1]\epsilon + [\ddot{x}_2 + x_1 + x_2]\epsilon^2 = 0.$$

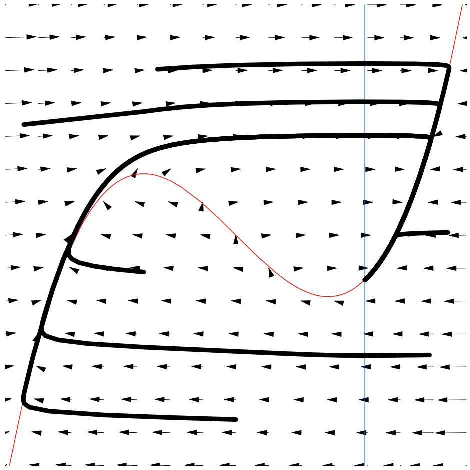

Figure 5: Phase portrait for $a > a_c$

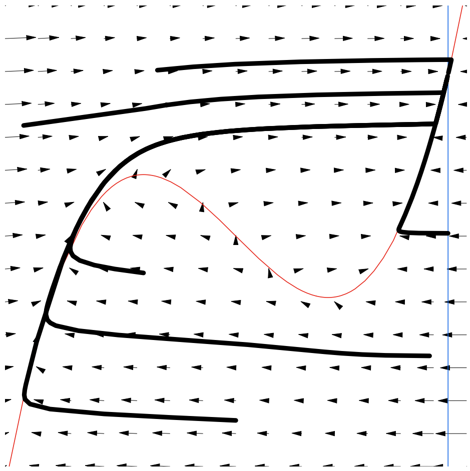

Figure 6: Phase portrait for $a >> a_c$

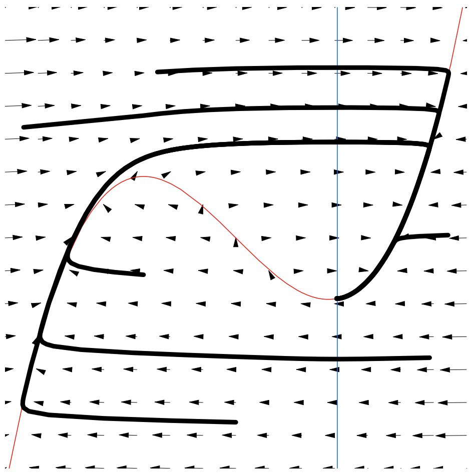

Figure 7: Phase portrait for a slightly greater than a_c

The key in using this is that for this sum to be equal to zero, each of the bracketed terms has to be equal to zero; in addition, each of the bracketed terms must satisfy the initial conditions. Solving for x_0 we have the diffeq $\ddot{x}_0 = -x_0$, which, given the initial conditions, is clearly seen to have the solution $x_0(t) = \cos t$. Our next differential equation has x_0 in it, so we write it as $\ddot{x}_1 + \cos t + x_1 = 0$. We know how to solve this sort of differential equation from last quarter, but Mathematica is good for this sort of thing. It gives the solution $x_1(t) = \cos(t) - \frac{1}{2}t\sin(t)$, (after running FullSimplify). Now we get to use this equation in our next differential equation, which is $\ddot{x} = \cos(t) - \frac{1}{2}t\sin(t) - \cos(t)$. The solution, which I got from mathematica, is $x_2(t) = \cos(t) - \frac{1}{8}t^2\cos(t) - \frac{3}{8}t\sin(t)$.

c) Does the perturbation solution contain secular terms (these are terms that grow without bound as $t \to \infty$)? Did you expect to see any? Why?

Yes, both x_1 and x_2 grow without bound as $t \to \infty$, because of the t and t^2 terms in their equations. We shouldn't expect to see any in a good aproximation to the solution, since the actual solution is not secular.