Hartle Solutions week 7
Problem 6.6

a) Writing out the differentials dt, dx, dy, dz in terms of dt', dx', dy', dz' and substituting into the standard line element for flat spacetime gives

$$d\tau^2 = -c^2\left(1+\frac{gx'}{c^2}\right)^2 dt'^2 + dx'^2 + dy'^2 + dz'^2 \,. \tag{1}$$

b) Expanding the cosh and sinh we have for small gt'/c,

$$\begin{aligned} t &\approx t' \\ x &\approx x' + \frac{1}{2}\, gt'^2 = x' + \frac{1}{2}\, gt^2 \end{aligned}$$

which is the transformation to an accelerated frame.

c) Clocks at rest in an accelerated frame have constant x'.

$$\begin{aligned} (d\tau)_{x'=0} &= dt' \\ (d\tau)_{x'=h} &= dt'\left(1+\frac{gh}{c^2}\right) \end{aligned}$$

so

$$(d\tau)_{x'=h} = (d\tau)_{x'=0}\left(1+\frac{gh}{c^2}\right) .$$

Thus, the clock higher up in the accelerated frame runs faster. This is an expression of the equivalence principle idea.

of a radioactive element with a decay time of 4 billion years were present to start, how much more of that element would be present at the center than the surface? Assume the density of the Earth is constant.

Solution: We give two different approaches to a solution:

First version: Suppose for simplicity that the density of the earth ρ is constant over its radius. The gravitational force on a particle of mas m at radius r is

$$F_r = -\frac{GmM(r)}{r^2} = -m\left(\frac{4}{3}\pi G\rho r\right) .$$

The gravitational potential difference between the center and the surface $R_\oplus$ is therefore

$$\Delta\Phi = \Phi(R_\oplus) - \Phi(0) = -\int_0^{R_\oplus} \frac{F_r}{m}\, dr = \frac{2}{3}\pi G\rho R_\oplus^2 = \frac{1}{2}\left(\frac{GM_\oplus}{R_\oplus}\right) .$$

Thus, (see useful constants)

$$\begin{aligned} \Delta\Phi/c^2 &= (.443 \text{ cm})/(2 \times 6.38 \times 10^8 \text{ cm}) \\ &= 3.47 \times 10^{-10} \end{aligned}$$

From (6.23) we can find the difference in elapsed times,

$$\Delta\tau_0 = \left(1 + 3.47 \times 10^{-10}\right)\tau_{R_\oplus} .$$

The abundance of a radioactive species will be

$$N \propto e^{-t/T}$$

where T is the decay time. Thus the ratio of abundances is

$$\frac{N_{\text{center}}}{N_{\text{surface}}} = \frac{e^{-(\tau_0/T)}}{e^{-(\tau_{R_\oplus}/T)}} \approx \exp\left(\frac{\tau_{R_\oplus}}{T}\frac{\Delta\Phi}{c^2}\right)$$

$$N_{\text{center}}/N_{\text{surf}} \approx 1 + \frac{5}{4} \times \left(3.47 \times 10^{-10}\right)$$

— a very small difference!

Second version:

We first calculate the gravitational potential difference between the center of the earth and its surface. Let the radius of the surface be $R_\oplus$ and the mass of the earth be $M_\oplus$. The potential difference is

$$\Delta\Phi \equiv \Phi\left(R_\oplus\right) - \Phi(0) = -\int_0^{R_\oplus} \vec{F} \cdot d\vec{r} \ . \tag{1}$$

Here $\vec{F}$ is the gravitational force *per unit mass, i.e.*

$$\vec{F} = -\frac{GM(r)}{r^2}\ \vec{e}_r$$

where $M(r)$ is the mass inside a radius r. Assuming a constant density $\rho_\oplus$

$$M = \frac{4}{3}\,\pi\rho_\oplus r^3 = M_\oplus\left(\frac{r}{R_\oplus}\right)^3 \ .$$

so that

$$\vec{F} = -\frac{GM_\oplus}{R_\oplus^2}\ \left(\frac{r}{R_\oplus}\right)\ \vec{e}_r \ .$$

Inserting in the integral (1) gives

$$\Delta\Phi = \frac{1}{2}\,\frac{GM_\oplus}{R_\oplus} \ .$$

The surface is at a *higher* in gravitational potential than the center. A clock at the surface therefore runs faster by a factor of

$$\left(1 + \frac{\Delta\Phi}{c^2}\right) = \left(1 + \frac{1}{2}\ \frac{GM_\oplus}{c^2 R_\oplus}\right) \approx 1 + 3.5\times 10^{-10} \ .$$

The rocks at the center are therefore younger by

$$\left(5\times 10^9\ \text{yr}\right)\left(3.5\times 10^{-10}\right) = 1.7\ \text{yr} \ .$$

The abundance of a radioactive element with a half-life $\tau_{1/2} = 4\times 10^9$yrs decays as

$$N = N_0\ e^{-t/\tau_{1/2}} \ .$$

There will be more of the element at the center than at the surface by

$$e^{\frac{+1.7yrs}{(4\times 10^9 yrs)}} \approx 1 + \frac{1.7}{4\times 10^9} \approx 1 + 4.3\times 10^{-10}$$

6-11. *[E] Aging goes on at a slower rate at the center of a spherical mass than on its surface. Estimate how much mass would need to be assembled in a radius of 10 km such that if you lived at the center for 1 year you would emerge 1 day younger than those who had stayed outside and far away?*

Solution: The gravitational potential difference between the center of a sphere of mass M and radius R and infinity is of order

$$\Delta\Phi \sim GM/R\,.$$

The difference in *rates* between a clock at the center and a clock far away is therefore of order

$$\frac{\Delta\Phi}{c^2} \sim \frac{GM}{Rc^2}\,.$$

For a clock to lag behind a clock at infinity by one day in one year, its rate must be slower in rate by 1/365. Thus,

$$\frac{GM}{Rc^2} \sim \frac{1}{365}\,.$$

To express this in solar masses, divide by $GM_\odot/c^2 = 1.5$ km to find

$$\frac{M}{M_\odot} \sim \frac{1}{365}\left(\frac{10\text{ km}}{1.5\text{ km}}\right) \sim .02\,.$$

That's about 20 times the mass of Jupiter!

Problem6.13

Solution: There are two effects: (1) time dilation, and (2) the gravitational effect on clocks. Working to $1/c^2$, and combining these effects, the proper time along any trajectory is

$$\tau = \int dt\,\left[1 - \frac{1}{c^2}\left(\frac{1}{2}V^2 - \Phi\right)\right]\,,$$

or

$$\tau = T - \frac{1}{c^2}\int dt\,\left[\frac{1}{2}V^2(t) - gh(t)\right]$$

since $\Phi = gh$. The first observer throws the clock upwards from $h = 0$. It reaches a maximum height $h_{\max} = 1/2\,g(T/2)^2 = (1/8)\,gT^2$. Thus,

$$h(t) = h_{\max} - \frac{1}{2}gt^2$$
$$V(t) = -gt$$

assuming $t = 0$ is the time the peak of the trajectory is reached. The elapsed time is $\tau = T - \Delta\tau$, where

$$\Delta\tau \equiv \frac{1}{c^2}\int_{-T/2}^{+T/2} dt \left[\frac{1}{2}\, g^2 t^2 - g h_{\max} + \frac{1}{2}\, g^2 t^2\right] = -\frac{1}{24}\left(\frac{gT}{c}\right)^2 T$$

For the second observer who holds the clock at rest, $\Delta\tau = 0$. For the third observer

$$V = h_{\max}/(T/2) \;=\; \frac{1}{4}\, gT$$

and

$$h(t) \;=\; h_{\max} - \frac{1}{4}\, gT|t|$$

$$\begin{aligned} \Delta\tau &= \frac{1}{c^2}\int_{-T/2}^{+T/2} dt \left[\frac{1}{32}\, g^2T^2 - g h_{\max} + \frac{1}{16}\, g^2T^2\right] \\ &= -\frac{1}{32}\left(\frac{gT}{c}\right)^2 T \end{aligned}$$

The longest proper time is registered by the path that is thrown because it is the path of a free particle.

Problem6.14

Solution: To order $1/c^2$ accuracy the proper time along any of these curves is given by (6.25) so

$$\Delta\tau = P - \frac{1}{c^2}\int_0^P dt\left[\frac{\vec{V}^2}{2} - \Phi\right]$$

in an inertial frame in which the center of the Earth is approximately at rest.

a) For a circular orbit of period P, $\Phi = -GM/R$ where R is related to P by Kepler's law $P^2 = (4\pi^2/GM)R^3$. Further, $V^2/R = GM/R^2$. The net result for the above integral is

$$\Delta\tau = P\left(1 - \frac{3}{2}\frac{GM}{Rc^2}\right)$$

which can be entirely expressed in terms of P and M using Kepler's law.

b) For a stationary observer $\vec{V} = 0$

$$\Delta\tau = P\left(1 - \frac{GM}{Rc^2}\right)$$

which is a longer proper time than a). Therefore, the circular orbit, although an extremal curve, is not a curve of *longest* proper time.

c) There is zero elapsed proper time. A circular orbit is not a curve of shortest proper time either.

d) If the particle is thrown radially outwards with the right velocity so that it returns in time P that is another curve of extremal proper time. [TEXT DELETED] Recall Problem 12.