

Non-Linear Dynamics Homework Solutions

Week 8

March 8, 2009

8.1.1 For the following prototypical examples, plot the phase portrait as μ varies.

a)

$$\begin{aligned}\dot{x} &= \mu x - x^2 \\ \dot{y} &= -y\end{aligned}$$

In order to do this we first find the nullclines. The x -nullclines are given by $x = 0$ and $x = \mu$ while the y -nullcline is given by $y = 0$. Our fixed points occur at intersection between x and y nullclines, giving us the origin and $(\mu, 0)$ as our fixed points. Next, we plot the nullclines and then determine the directions of the flow along the nullclines by considering various values for each region. From this point we can get a pretty good idea of what kind of fixed points we have and how trajectories are going to behave. We could also find the Jacobian matrices to do much of this, by finding eigenvalues and stabilities at the fixed points. See Figure 1 for phase portraits for μ less than, equal to and greater than zero are given.

b)

$$\begin{aligned}\dot{x} &= \mu x - x^3 \\ \dot{y} &= -y\end{aligned}$$

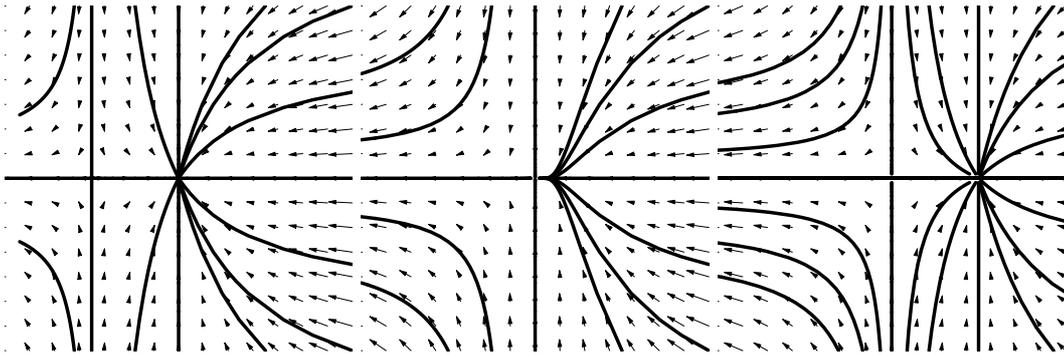


Figure 1: Phase Portraits for $\mu < 0$, $\mu = 0$ and $\mu > 0$, from left to right

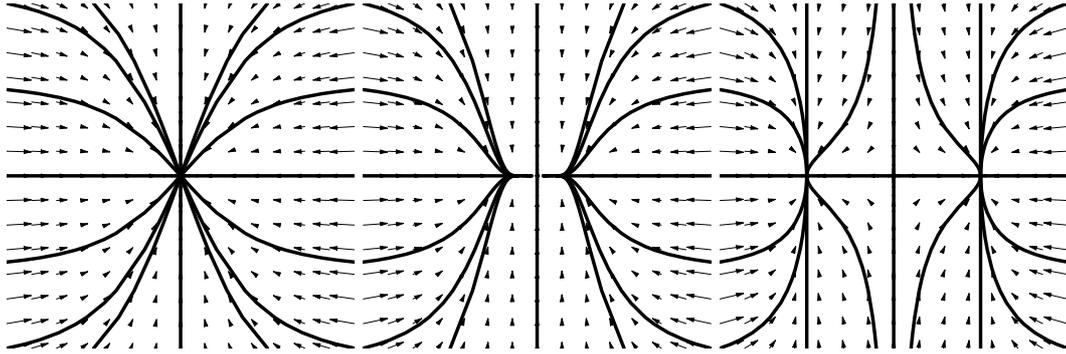


Figure 2: Phase Portraits for $\mu < 0$, $\mu = 0$ and $\mu > 0$, from left to right

As in part (a), we find our x -nullcline to be given by $x = 0$ and $x = \pm\sqrt{-\mu}$ and our y -nullcline to be $y = 0$. From this we conclude that our bifurcation occurs at $\mu = 0$, so we show phase portraits for values of μ to either side of $\mu = 0$ (see Figure 2). As above, to make these plots, first graph the nullclines, then determine the flow along them, then fill in the rest of the portrait around them in, using the linearization of the fixed points to determine eigenvectors and stabilities if necessary.

8.1.7 Find and classify all bifurcations of the system

$$\begin{aligned}\dot{x} &= y - ax \\ \dot{y} &= -by + \frac{x}{1+x}\end{aligned}$$

As with most problems in this subject, we begin by finding our x -nullcline to be given by $y = ax$ and our y -nullcline to be given by $y = x/(b(1+x))$. The intersections between these are the origin and the point $(1/ab - 1, 1/b - a)$. Next, we compute the Jacobian to be

$$J = \begin{pmatrix} -a & 1 \\ \frac{1}{(1+x)^2} & -b \end{pmatrix}$$

For the origin, $T = -(a+b)$ and $D = ab - 1$ and we find that $T = -(a+b)$ and $D = ab - (ab)^2$ for the other fixed point. Note that when either of a or b is zero, the origin becomes the only fixed point, but not through a collision, but through a process by which the second fixed point explodes to infinity in an ω - \blacklozenge ski slope bifurcation (see an earlier homework assignment).

To find our more traditional kinds of bifurcations, we think about these fixed points and what sort of trace determinant transitions we can have. Doing this we find that if we increase the value of ab from just below 1 to just above, the sign of D for the origin changes. Note also, that when $ab = 1$, our second fixed point merges with the origin. As it passes by the origin changes stability. Thus we conclude that along the curve $a = 1/b$ in (a, b) space, we have transcritical bifurcations occurring.

Next we find that for the origin $T^2 - D = (a - b)^2 + 1$, which is always positive, for obvious reasons. Doing the same for our other fixed point we find that $T^2 - 4D = (a - b)^2 + 4(ab)^2$, which also is always positive. What this tells us is that whenever D is positive for one of our fixed points, that fixed point is a node and not a spiral.

8.2.1 Consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. Find the curves in (μ, a) space at which Hopf bifurcations occur.

We start by writing this second order differential equation as a first order system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= a - x - \mu(x^2 - 1)y\end{aligned}$$

From this we find the x -nullcline to be $y = 0$ and the y -nullcline to be $y = (a - x)/(\mu(x^2 - 1))$. Intersections therefore happen iff $x = a$. Next we find the Jacobian to be

$$J = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu yx & -\mu(x^2 - 1) \end{pmatrix}$$

which when evaluated at our only fixed point $(a, 0)$ gives us $T = -\mu(a^2 - 1)$ and $D = 1$. For a Hopf bifurcation to occur, the trace must switch signs, and must at the bifurcation be equal to zero. Solving for $T = 0$ in terms of μ and a we find that $T = 0$ when $\mu = 0$ or $a = \pm 1$. These lines aren't very curvy, but they are the curves we're looking for.

8.2.8 (Predator Prey Model) Consider the model

$$\begin{aligned}\dot{x} &= x[x(1-x) - y] \\ \dot{y} &= y(x-a)\end{aligned}$$

a) Sketch nullclines in Quadrant I.

We find x -nullclines (in red) at $x = 0$ and $y = x(1-x)$ and y -nullclines (in blue) at $y = 0$ and $x = a$. See Figure 3.

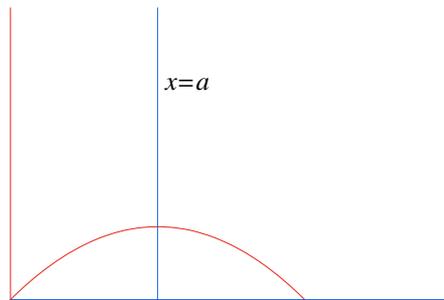


Figure 3: Nullclines in the 1st Quadrant

b) The origin and the point $(1, 0)$ are clearly seen to be fixed points, given the factored forms of the equations above. If, however, $\dot{y} = 0$ but $y \neq 0$ then $x = a$ so $y = a(1-a)$. Thus $(a, a - a^2)$ is a fixed point of the system.

To classify these fixed points, we compute the Jacobian

$$J = \begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - a \end{pmatrix}$$

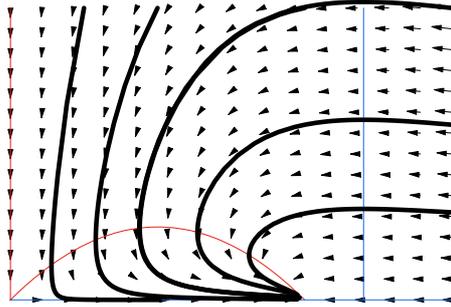


Figure 4: Phase portrait for $a > 1$

and find that $T = 3x - 3x^2 - y - a$ and $D = (2x - 3x^2 - y)(x - a) + yx$. By plugging in the various values of x and y corresponding to each of the fixed points we find the following table of traces and determinants, as well as classifications corresponding to various values of a .

	$(0, 0)$	$(1, 0)$	$(a, a - a^2)$
T	$-a$	$-a$	$a(1 - 2a)$
D	0	$a - 1$	$a^2(1 - a)$
$T^2 - 4D$	a^2	a^2	$a^2(4a^2 - 3)$

From this handy table, it is easy to see that the origin will always be marginally somewhere between a stable node and a saddle; $(1, 0)$ will start off as a saddle, but as a is varied above 1, will become a stable node, but will never be a spiral since $T^2 - 4D$ is always positive for it; $(a, a - a^2)$ will be by far the most interesting of the fixed points. First note that D is positive for all $a < 1$ and negative otherwise, so as it passes through the point $(1, 0)$ as a is varied past 1, it changes from being a node to a saddle. Hence $a_t = 1$ corresponds to a transcritical bifurcation. To see what else is going on with this fixed point, note that for $a < \sqrt{3/4}$, our third fixed point is a spiral. What's more, when $a < 1/2$ then $T > 0$, but $T < 0$ when $a > 1/2$, so the spiral goes from being unstable to stable as we vary a past the critical point $a_c = 1/2$.

- c) Sketch the phase portrait for $a > 1$ and show that the predators go extinct.
See Figure 4 for a numerical plot given that $a = 1.2$.
- d) Show that a Hopf bifurcation occurs at $a_c = 1/2$. Is it subcritical or supercritical?
We have already shown that the spiral at $(a, a - a^2)$ goes from being stable to unstable as we vary a from $a > 1/2$ to $a < 1/2$. What remains to be shown is that we do actually get a limit cycle out of this bifurcation. Figure 5 shows that as this change occurs, a stable limit cycle does indeed arise.
From this plot we deduce that we have a supercritical Hopf bifurcation at $a_c = 1/2$.
- e) Estimate the frequency of limit cycle oscillations for a near a_c .
We use the second rule of thumb given on page 251 of Strogatz, which states that the frequency of the limit cycle is given approximately by the imaginary part of the eigenvalue of the linearization matrix at the moment of bifurcation. Since the eigenvalue of a 2×2 matrix is $\sqrt{T^2 - 4D}/2$, we compute the frequency to be $2^{-3/2}$.

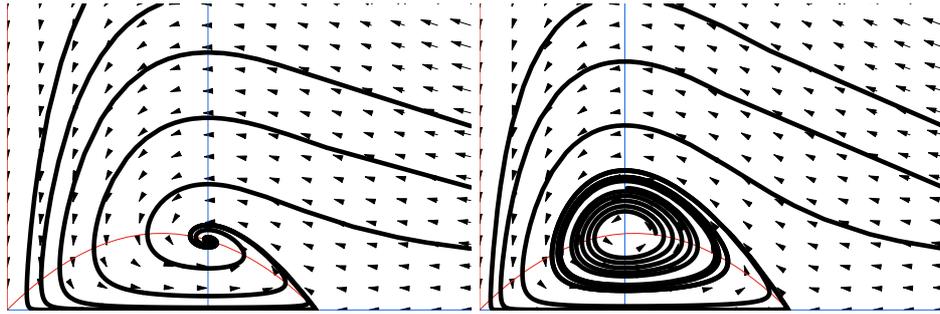


Figure 5: Phase portraits for $a > 1/2$ and $a < 1/2$ on the left and right respectively

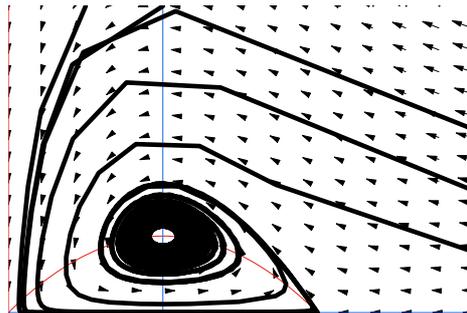


Figure 6: Phase portrait for $a = a_c$

f) Sketch all of the topologically different phase portraits for $0 < a < 1$.

We have already shown graphs in the cases when $a < a_c$ and $a > a_c$, so to complete things, we show a graph when $a = a_c$ (See figure 6). What we find is that a critical slowing down is occurring. The graph is somewhat deceiving; it appears that there might actually be a limit cycle here, but that is only an artifact of how slow the system is moving. To give you a feel for it, for the graph in Figure 5 with $a > 1/2$, I did a numerical evaluation for t between 0 and 100, with $a = .65$ and everything made it to the center. For Figure 6, I had to evaluate between 0 and 1000, and still the solutions did not reach the center. This large evaluation range is also why the solutions look like connect-the-dots early on in the trajectories.

8.3.1 (Brusselator) The kinetics of the Brusselator are given by

$$\begin{aligned}\dot{x} &= 1 - (b + 1)x + ax^2y \\ \dot{y} &= bx - ax^2y\end{aligned}$$

where $a, b > 0$ are parameters and $x, y \geq 0$ are dimensionless concentrations.

a) Find all the fixed points, and use the Jacobian to classify them.

We find that our x -nullcline is

$$y = \frac{(b + 1)x - 1}{ax^2}$$

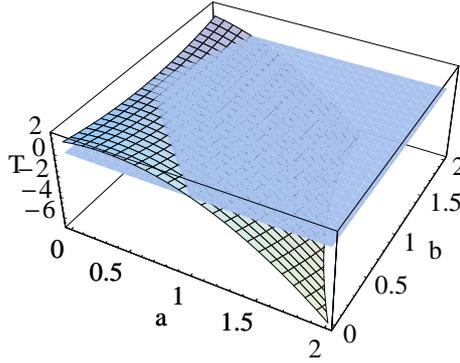


Figure 7: Plot of T vs a and b for Problem 8.3.1

and our y -nullcline is given by

$$y = \frac{bx}{ax^2}.$$

These intersect only when $(b+1)x - 1 = bx$, which happens only if $x = 1$. Therefore, our only fixed point is at $(1, b/a)$.

We find the Jacobian to be given by

$$J = \begin{pmatrix} -(b+1) + 2axy & ax^2 \\ b - 2axy & -ax^2 \end{pmatrix}.$$

When evaluated at our fixed point this goes to

$$J|_{x^*} = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}.$$

From this we find that

$$T = b - a - 1,$$

$$D = a^2,$$

$$T^2 - 4D = (b - a - 1)^2 - 4a^2$$

From these calculations we see that we have a change in stability along the line $b = a + 1$ in (a, b) phase space. Above this line we have instability (since in this region $T > 0$), and below it stability. Furthermore we conclude that the determinant is always positive and consequently, the fixed point will never be a saddle. Now if we plot $T^2 - 4D$ as a function of a and b (See Figure 7), then we can see that in fact we do have node and spiral behaviour. Furthermore, these two behaviours are separated from each other by the curve defined by $T^2 - 4D = 0$, which can be expressed in terms of b vs. a by setting the expression equal to zero and using the quadratic formula, which leads to the solutions $b = 1 - a$, and $b = 1 + 3a$. Putting all of this information together, we get the following phase space diagram in Figure 8.

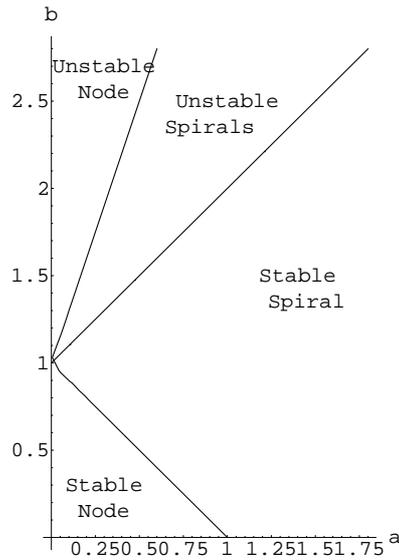


Figure 8: Stability of fixed points in (a, b) phase space

- b) Sketch the nullclines, and thereby construct a trapping region for the flow.

Figure 9 has a plot of the nullclines against the vector field. We know that since we only have one fixed point (intersection of nullclines), the nullclines will divide the plane into regions I-IV. Throughout region I, the flow will be down and to the right, in region II, down and to the left, and so on. Since in region IV we have all vectors pointing up and to the right, a vertical line through this region will witness that all vectors lying along it point in toward the fixed point. We want our line to be given by $x = c$ for some constant c which is inbetween 0 and the x -intercept of the x -nullcline. For simplicity, we shall choose c to be precisely halfway inbetween. We know that along the line $y = 0$, all trajectories are pointing up (since in region IV all lines point up and to the right, in region III they all point up and to the left, and along the boundary they all point up). Consequently, this will make a good bottom trapping region boundary.

Our next task (and boy is it a tricky one!), is to close off the trapping region with boundaries in region I. The basic idea is that from wherever the line $x = c$ meets up with the y nullcline, we need to go up a little bit higher (this is okay, since flows point to the right in both regions IV and I), and then to move over to the right in a horizontal line (okay since all vectors point down and to the right here) which shall end at $x = 1$. From this point the tricky part begins. We need from this point to come up with a line which starts at this afore mentioned point, and works it's way gently down until it meets the y -nullcline. At this point we can draw a line from this point straight down to the x -axis to meet up with our bottom region boundary. The trick is to find the right slope for the line mentioned.

Let's dig in. First we find that the y -intercept of the x -nullcline is at $x = 1/(b + 1)$ (we find this by setting the x -nullcline equal to zero and solving for x). Thus we want to take our left hand side trapping region boundary as $x = 1/(2b + 2)$ so that the boundary is

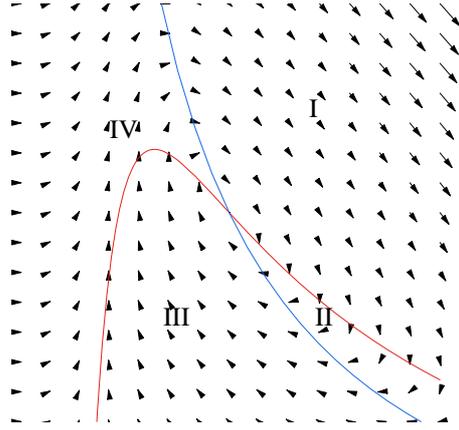


Figure 9: Plot of the nullclines for Problem 8.3.1

in between 0 and the x intercept. Next we note that this line intercepts the x -nullcline at $y = 2(b^2 + b)/a$. For safe measure, we can take our line all the way up to $y = 3(b^2 + b)/a$. Next we move along the line $y = 3(b^2 + b)/a$ until we reach the point $x = 1$.

Now comes the tricky part. We have to find a line starting from this point we've left off at and descending down to the x axis, so that its slope is greater than the "slopes" of the vectors. Now, the vectors have slopes given by

$$m = \frac{\dot{y}}{\dot{x}} = \frac{bx - ax^2y}{1 - (b+1)x + ax^2y}.$$

This is ugly. It's not at all clear what the maximum of this value is going to be over the region we are interested in, and it's especially hard when we still have everything in terms of a and b . One technique we can use is to modify m by taking out or adding terms so that it becomes simpler but is always greater than the actual value of m . In this spirit, we note that

$$\begin{aligned} m &= -\frac{ax^2y - bx}{ax^2y - bx - (x-1)} \\ &< -\frac{ax^2y - bx}{ax^2y - bx} \\ &= -1, \end{aligned}$$

where the inequality holds for our region of interest because if we add $x - 1$ (a positive number for our region of interest) to the denominator, the absolute value of the resulting fraction gets smaller, so the negative of it will be bigger. This is awesome, because we can just set the slope of the line equal to $-1/2$. Consequently, the line that we want is going to be $y = -(x-1)/2 + 1/(2b+2) + 1$ so that when $x = 1$ y is equal to supposed to. This was the only tricky bit to getting an outer trapping region, so we are done.

- c) Show that a Hopf bifurcation occurs at some parameter value $b = b_c$, where b_c is to be determined.

We have already shown that when $b = a + 1$, we have a switch in stability of our fixed point. As soon as the fixed point becomes unstable, we will be able to get the inner boundary for the trapping region, implying by the Poincaré-Bendixon theorem that there will be a stable limit cycle.

- d) Does the limit cycle exist for $b > b_c$ or $b < b_c$? Explain, using the Poincaré-Bendixon theorem.

Looking at Figure 8, we see that when $b > b_c = a + 1$, we are in the unstable region, implying that the limit cycle exists under these conditions.

- e) Find the approximate period of the limit cycle for $b \approx b_c$.

The period of the limit cycle will be approximately equal to the imaginary part of the eigenvalue at the moment of bifurcation. At this moment, the imaginary part of the eigenvalue is going to be $\sqrt{T^2 - 4D}/2 = \sqrt{(b - a - 1)^2 - 4a^2}$.

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